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# **Restricted Choice for Change and Reasoning**

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# Choice Functions

## Mathematics

Let  $\mathbb{S} \subseteq \mathcal{P}(A)$  be a set of sets.

$C : \mathbb{S} \rightarrow A$  is a **choice function** if

$X \neq \emptyset$  implies  $C(X) \in X$

$$\mathbb{S} = \{ \{1, 4, 7\}, \{9\}, \{1, 7\} \}$$

$$C(\{1, 4, 7\}) = 1$$

$$C(\{9\}) = 9$$

$$C(\{1, 7\}) = 7$$

## Social Choice Theory

Let  $\mathbb{S} \subseteq \mathcal{P}(A)$  be a set of sets.

$C : \mathbb{S} \rightarrow \mathcal{P}(A)$  is a **choice function** if

$C(X) \subseteq X$ , and

$X \neq \emptyset$  implies  $C(X) \neq \emptyset$

$$C(\{1, 4, 7\}) = \{4, 7\}$$

$$C(\{9\}) = \{9\}$$

$$C(\{1, 7\}) = \{1\}$$

# Applications of Choice Functions

- Set theory
  - ▶ Axiom of Choice
    - (AC) For any set of sets  $\mathcal{S}$ , there exists a choice function defined on  $\mathcal{S}$
  - ▶ Battery of equivalences to (AC), e.g., well-ordering theorem
- Rationality in economics
- Voting theory
- Knowledge Representation and Reasoning
  - ▶ Selection functions
  - ▶ Representation of extra-logical information

# Representation of Choice Functions (via Orders)

Amarty Sen. *Choice Functions and Revealed Preferenc.* The Review of Economic Studies 15(1): p. 307–317, 1971

From  $C : \mathbb{S} \rightarrow \mathcal{P}(A)$  to  $\leq_C \subseteq A \times A$ :

$$x \leq_C y \quad \text{if and only if} \quad x \in C(S) \text{ for some } S \in \mathbb{S} \text{ with } x, y \in S$$

From  $\leq \subseteq A \times A$  to  $C_{\leq} : \mathbb{S} \rightarrow \mathcal{P}(A)$ :

$$C_{\leq}(S) = \min(S, \leq) = \{s \in S \mid s \leq q \text{ for all } q \in S\}$$

**Theorem (Sen, 1971).** Let  $\mathbb{S} = \mathcal{P}(A)$ . There are properties<sup>1</sup>  $\alpha$  and  $\gamma$  s.t.

$$C = C_{\leq} \text{ for some } \leq \text{ if and only if } C \text{ satisfies } \alpha \text{ and } \gamma$$

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<sup>1</sup>For finite  $\mathbb{S}$ :  $\alpha$  is  $C(S_1 \cup S_2) \subseteq C(S_1) \cup C(S_2)$  and  $\gamma$ :  $C(S_1) \cap C(S_2) \subseteq C(S_1 \cap S_2)$

# Buying Snacks

A universe of snacks:

$$A = \{\text{liquorice, nachos, pretzels, dips, chillies}\}$$

Potential snack options for a lavish evening with friends:

$$S = \{\text{nachos, liquorice, pretzels}\}$$

'Please bring one of them'

$$C(S) = \{ \text{nachos} \}$$

Order  $\leq$  that supports C:


$$C(S) = \min(S, \leq)$$

## Buying Snacks II

$$A = \{ \text{liquorice, nachos, pretzels, dips, chillies} \}$$

$$S = \{ \text{nachos, liquorice, pretzels} \}$$

$$C(S) = \{ \text{nachos} \}$$

Available packages in the supermarket:

$$\mathbb{E} = \{ \{ \text{nachos, dips} \}, \{ \text{pretzels} \} \}$$

Consequently, the choice  $C(S) = \{ \text{nachos} \}$  is **impossible**.

Choices are restricted — You can't always get what you want

# Restrict Choice Functions

## Restricted Choice Structure

$R = \langle A, \mathbb{S}, \mathbb{E} \rangle$  with

- $\mathbb{S} \subseteq \mathcal{P}(A)$
- $\mathbb{E} \subseteq \mathcal{P}(A)$  and  $\mathbb{E} \subseteq \mathbb{S}$

## Choice Function for $R$

$C : \mathbb{S} \rightarrow \mathbb{E}$  such that

- $C(S) = A \in \mathbb{E}$  with  $A \subseteq S$ ,  
if such an  $A \in \mathbb{E}$  exists.

## Fallback

$K \in \mathbb{E}$  is fallback of  $C : \mathbb{S} \rightarrow \mathbb{E}$  if

- $C(S) = K$  whenever  
there is no  $A \in \mathbb{E}$  with  $A \subseteq S$ .

$A = \{ \text{liquorice, nachos,} \\ \text{pretzels, dips, chillies} \}$

$\mathbb{S} = \mathcal{P}(A)$

$\mathbb{E} = \{ \{ \text{nachos, dips} \}, \{ \text{pretzels} \} \}$

$K = \{ \text{nachos, dips} \}$

$S = \{ \text{nachos, liquorice, pretzels} \}$

$C(S) = \{ \text{pretzels} \}$

$S' = \{ \text{liquorice} \}$

$C(S') = \{ \{ \text{nachos, dips} \} \}$

**Does for every  $R = \langle A, \mathbb{S}, \mathbb{E} \rangle$  a choice function exist at all?**

**If yes, how can we represent such a choice function?**

# Orders over the Universe: too limited

$R = \langle A, \mathbb{S}, \mathbb{E} \rangle$  and  $K$  with:

$$A = \{ \text{liquorice, nachos, pretzels, dips, chillies} \}$$

$$\mathbb{S} = \mathcal{P}(A)$$

$$\mathbb{E} = \{ \{ \text{nachos, dips} \}, \{ \text{pretzels} \} \}$$

$$K = \{ \text{nachos, dips} \}$$

**Claim:** no order  $\leq \subseteq A \times A$  encodes a choice function for  $R$ .

From  $C(\{ \text{nachos, dips} \}) = \{ \text{nachos, dips} \}$  obtain:

$$\text{nachos} \leq \text{dips}$$

$$\text{dips} \leq \text{nachos}$$

So we have:

$$\min(\{ \text{nachos} \}, \leq_C) = \{ \text{nachos} \} \neq C(\{ \text{nachos} \}) = \{ \text{nachos, dips} \}$$

# Choice Functions via Linear Orders of Sets

## Linear Choice Function

Let  $\ll \subseteq \mathcal{E} \times \mathcal{E}$  be a well-founded linear order over some set  $\mathcal{E} \subseteq \mathbb{E}$  and let  $\{K\} = \min(\mathcal{P}(A), \ll)$ .

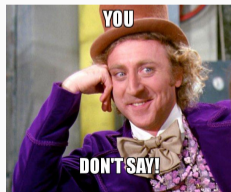
A function  $\nabla : \mathcal{S} \rightarrow \mathbb{E}$  with

$$\nabla(S) = \min(S, \ll) = \begin{cases} E & \text{if } \{E\} = \min(\{E' \mid E' \subseteq S\}, \ll) \\ K & \text{otherwise} \end{cases}$$

is called a *linear choice function*. We also write  $\nabla_{\ll}$  to make  $\ll$  explicit and denote  $K$  as the *fallback value* of  $\nabla$ .

**Theorem** Assume the Axiom of Choice.

For every  $R = \langle A, \mathcal{S}, \mathbb{E} \rangle$  there is a linear choice function.



$$\nabla_{\ll}(S) = \min(S, \ll) = \begin{cases} E & \text{if } \{E\} = \min(\{E' \mid E' \subseteq S\}, \ll) \\ \min(\mathcal{P}(A), \ll) & \text{otherwise} \end{cases}$$

Example  $R = \langle A, \mathbb{S}, \mathbb{E} \rangle$ :

Linear order  $\ll$ :

$A = \{ \text{liquorice, nachos, pretzels, dips, chillies} \}$	$\{ \text{pretzels} \}$
$\mathbb{S} = \mathcal{P}(A)$	
$\mathbb{E} = \{ \{ \text{nachos, dips} \}, \{ \text{pretzels} \} \}$	$\{ \text{nachos, dips} \}$

Choice function:

$$\nabla_{\ll}(S) = \begin{cases} \{ \text{pretzels} \} & \text{if } \text{pretzels} \in S \text{ and} \\ & \{ \text{nachos, dips} \} \not\subseteq S \\ \{ \text{nachos, dips} \} & \text{otherwise} \end{cases}$$

$$\nabla_{\ll}(S) = \min(S, \ll) = \begin{cases} E & \text{if } \{E\} = \min(\{E' \mid E' \subseteq S\}, \ll) \\ \min(\mathcal{P}(A), \ll) & \text{otherwise} \end{cases}$$

Example  $R = \langle A, \mathbb{S}, \mathbb{E} \rangle$ :

Linear order  $\ll$ :

$$A = \mathbb{R}$$

$$\mathbb{S} = \mathcal{P}(A)$$

$$\mathbb{E} = \{\{n\} \mid n \in \mathbb{N}_0 \setminus \{2\}\} \cup \{\{1, 2\}\}$$

Choice function:

$$\nabla_{\ll}(S) = \begin{cases} \{1, 2\} & \text{if } 0 \notin S, \{1, 2\} \subseteq S \\ \{\min(S \cap \mathbb{N}_0)\} & \text{if } S \cap (\mathbb{N}_0 \setminus \{2\}) \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}$$

$$\nabla_{\ll}(\{\pi\}) = \{0\}, \nabla_{\ll}(\{\pi, 1, 2\}) = \{1, 2\}, \nabla_{\ll}(\mathbb{R}) = \{0\}$$



**Can we say something interesting about linear choice functions?**

- **Axiomatics**
- Revision
- Reasoning and Argumentation

# Axiomatization

**Theorem** Assume the Axiom of Choice and let  $R = \langle A, \mathbb{S}, \mathbb{E} \rangle$  be union-closed<sup>2</sup>.

A function  $\nabla : \mathbb{S} \rightarrow \mathbb{E}$  is linear choice function with fallback  $K$  if and only if:

- (SS1)  $\nabla(S) \subseteq S$  or  $\nabla(S) = K$ . (Relative Success)
- (SS2) If  $K \subseteq S$ , then  $\nabla(S) = K$ . (Idempotence)
- (SS3) If  $\nabla(S_1) \subseteq S_2$  and  $\nabla(S_2) \subseteq S_1$ , then  $\nabla(S_1) = \nabla(S_2)$ . (Right-Reciprocity)
- (SS4) If  $\nabla(S_1) \subseteq S_1$  and  $S_1 \subseteq S_2$ , then  $\nabla(S_2) \subseteq S_2$ . (Monotonicity)
- (SS5) If  $\nabla(S_i \cup S_{i+1}) = S_i$  for  $0 \leq i \leq n$ ,  
then  $S_0 \neq S_n$  implies  $\nabla(S_0 \cup S_n) \neq S_n$ . (Suzumura)
- (SS6) If  $\nabla(S_1 \cup S_2) = S_3$ , then  $\nabla(S_1 \cup S_3) = S_3$ . (Prevalence)

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<sup>2</sup>For all  $S_1, S_2 \in \mathbb{S}$  holds  $S_1 \cup S_2 \in \mathbb{S}$

Can we say something interesting about linear choice functions?

- Axiomatics
- **Revision**
- Reasoning and Argumentation

# Choice-Based Belief Revision

Let  $\mathbb{S} \subseteq \mathcal{P}(A)$ . A change operator for  $\mathbb{S}$  is a function  $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ .

Typical reading:

*initial beliefs*  $\odot$  *new information* = *new beliefs*

## Definition

$\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$  is called *choice-based* if for each  $K \in \mathbb{S}$  there is a choice function  $C_K$  for some restricted choice structure  $R_K = \langle A, \mathbb{S}, \mathbb{E}_K \rangle$  with  $K \in \mathbb{E}_K$  such that:

$$K \odot S = C_K(S)$$

We additionally say that  $\odot$  is *linear*, if each  $C_K$  is a linear choice function for  $R$ .

Example  $R_K = \langle A, \mathbb{S}, \mathbb{E}_K \rangle$ :

$$A = \mathbb{R}$$

$$\mathbb{S} = \mathcal{P}(A)$$

$$\mathbb{E}_K = \{ \{n\} \mid n \in \mathbb{N}_0 \} \cup \{ \{K\} \}$$

Choice function for every  $K$ :

$$\nabla_K(S) = \begin{cases} K & \text{if } K \subseteq S \\ \{ \min(S \cap \mathbb{N}_0) \} & \text{if } S \cap \mathbb{N}_0 \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

Linear order  $\ll_K$  for  $\nabla_K$ :

$$\begin{array}{c} \vdots \\ | \\ \{2\} \\ | \\ \{1\} \\ | \\ \{0\} \\ | \\ \{K\} \end{array}$$

Examples:  $K \circlearrowleft \{\pi\} = K$  or  $K \circlearrowleft \{1, 7\} = \{1\}$

# Representation Theorem

**Theorem** Assume the Axiom of Choice, let  $\mathbb{S}$  be union-closed, and let  $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$  be a change operator for  $\mathbb{S}$  (over  $A$ ).

The operator  $\odot$  is a linear choice-based change operator if and only if

(LCR1)  $K \odot S \subseteq S$  or  $K \odot S = K$ . (Relative Success)

(LCR2) If  $K \subseteq S$ , then  $K \odot S = K$ . (Idempotence)

(LCR3) If  $K \odot S_1 \subseteq S_2$  and  $K \odot S_2 \subseteq S_1$ , then  $K \odot S_1 = K \odot S_2$ . (Right-Reciprocity)

(LCR4) If  $K \odot S_1 \subseteq S_1$  and  $S_1 \subseteq S_2$ , then  $K \odot S_2 \subseteq S_2$ . (Monotonicity)

(LCR5) If  $K \odot (S_i \cup S_{i+1}) = S_i$  for  $0 \leq i \leq n$ , then  $S_0 \neq S_n$   
implies  $K \odot (S_0 \cup S_n) \neq S_n$ . (Suzumura)

(LCR6) If  $K \odot (S_1 \cup S_2) = S_3$ , then  $K \odot (S_1 \cup S_3) = S_3$ . (Prevalence)

# Meaning of Restricted Choice

Typical reading for  $\odot : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{S}$ :

*initial beliefs*  $\odot$  *new information* = *new beliefs*

The output is restricted to elements of  $\mathbb{E}$ .

KR Interpretations:

- Mismatch between logic and real world
- Limited mental capacities/Unawareness

Classical revision theory cannot express this. Compare to credibility-limited revision:

- If  $\{\text{nachos}, \text{dips}\}$  is credible, then also  $\{\text{nachos}\}$  and  $\{\text{dips}\}$
- In general, this is not the case in restricted choice

# Theoretical Relevance

## **Observation (Sauerwald and Thimm, KR2025)**

*There are settings where there is no AGM revision operator.*

Choice-Based Belief Revision is a basic theory of change:

- Choice-Based Belief Revision subsumes AGM revision
- Linear change operators are available in any setting!

**Can we say something interesting about linear choice functions?**

- Axiomatics
- Revision
- **Reasoning and Argumentation**

# Non-Monotonic reasoning based on restricted choice

Define  $\vdash \subseteq \mathbb{S} \times \mathbb{S}$  for a choice function  $C$ :

$$S_1 \vdash S_2 \quad \text{if } C(S_1) \subseteq S_2$$

Axiomatics of the 'linear choice version'<sup>3</sup>:

(LCE1)  $S \vdash S$  or  $S \vdash K$ . (Relative Reflexivity)

(LCE2) If  $K \vdash S$ , then  $K \bowtie S$ . (Idempotence)

(LCE3) If  $S_1 \vdash S_2$  and  $S_2 \vdash S_1$ , then  $S_1 \bowtie S_2$ . (Right-Reciprocity)

(LCE4) If  $S_1 \vdash S_1$  and  $S_1 \subseteq S_2$ , then  $S_2 \vdash S_2$ . (Monotone Reflexivity)

(LCE5) If  $S_i \cup S_{i+1} \bowtie S_i$  for  $0 \leq i \leq n$ , then  $S_0 \neq S_n$  implies  $S_0 \cup S_n \bowtie S_n$ . (Suzumura)

(LCE6) If  $S_1 \cup S_2 \bowtie S_3$ , then  $S_1 \cup S_3 \bowtie S_3$ . (Prevalence)

<sup>3</sup> $S_1 \bowtie S_2$  if for all  $S \in \mathbb{S}$  holds:  $C(S_1) \vdash S$  if and only if  $S_2 \subseteq S$ .

# Choice-based Argumentation Semantics

Abstract Argumentation Framework (AF):

$$F = \langle A, R \rangle$$

where  $A$  is a set and  $R \subseteq A \times A$  (attack relation).

We are looking for extensions  $E \in \mathcal{P}(A)$  ('sets of jointly acceptable arguments')

## Definition

A *choice-based extension semantics* is a function  $\Pi$  that maps each AF  $F = \langle A, R \rangle$  to a choice function  $\Pi_F : \mathcal{P}(A) \rightarrow \mathbb{E}_F$  for some choice structure  $R_F = \langle A, \mathcal{P}(A), \mathbb{E}_F \rangle$  where  $\mathbb{E}_F$  is non-empty.

We say that  $\Pi$  is *linear*, if every  $\Pi_F$  is a linear choice function with fallback  $K_F$  for some  $K_F \in \mathbb{E}_F$ .

# Axiomatics: Linear Choice-based Extension Semantics

- (LCA1)  $\Pi_F(E) \subseteq E$  or  $\Pi_F(E) = K_F$ . (Relative Reflexivity)
- (LCA2) If  $K_F \subseteq \Pi_F(E)$ , then  $\Pi_F(E) = K_F$ . (Idempotence)
- (LCA3) If  $\Pi_F(E_1) \subseteq E_2$  and  $\Pi_F(E_2) \subseteq E_1$ , then  $\Pi_F(E_1) = \Pi_F(E_2)$ . (Right-Reciprocity)
- (LCA4) If  $\Pi_F(E_1) \subseteq E_1$  and  $E_1 \subseteq E_2$ , then  $\Pi_F(E_2) \subseteq E_2$ . (Monotonicity)
- (LCA5) If  $\Pi_F(E_i \cup E_{i+1}) = E_i$  for  $0 \leq i \leq n$ , then  $E_0 \neq E_n$  implies  $\Pi_F(E_0 \cup E_n) \neq E_n$ . (Suzumura)
- (LCA6) If  $\Pi_F(E_1 \cup E_2) = E_3$ , then  $\Pi_F(E_1 \cup E_3) = E_3$ . (Prevalence)

# Remarks

Linear choice-based non-monotonic reasoning:

- It makes sense

Linear choice-based extension semantics:

- The fallback value makes no sense

Reasonable fallback value  $K = \emptyset$  ( $\approx$  'no arguments together are reasoning').

However, such a fallback value would dominate everything:

(LCA2) If  $K_F \subseteq \Pi_F(E)$ , then  $\Pi_F(E) = K_F$ . (Idempotence)

## Content

- Choice Functions
- Snack example
- Restricted Choice
- Choice Functions for Restricted Choice
- Linear Choice Functions
- Axiomatization
- Importance for Belief Revision
- NMR and Argumentation

## Future work

- More application in Belief Revision
- More application to NMR
- More application in Argumentation
- Different Fallbacks

Thanks for your attention!