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New Results on Preferential Reasoning

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Joint work with Arne Meier and Juha Kontinen

Introduction and Motivation

1. Logic Background

2. Preferential Logics

3. Axiomatics

4. Complexity

Conclusion

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1\. Logic Background

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Conclusion

combine

Preferential Reasoning

(injection of extra-logical information into reasoning)

and

Team-based Reasoning

(reasoning in presence of plurality of objects)

Motivation I: Reasoning over different data sources

Living in a world of many sensors:

- Provide large amounts of data
- Different reliability

One scenario:

- Set of sensors $\mathcal{S} = \{s_1, s_2, \ldots\}$
- Each sensor s has a database $\ell(s)$
- $\prec$ ordering over the sensors; meaning:

$$s_1 \prec s_2 \text{ if } s_1 \text{ is strictly more reliable then } s_2$$

Research question:

- Sensor s supports α, if α holds in $\ell(s)$
- Investigate the following kind of reasoning

$$\alpha \mid\!\sim \beta \text{ if } \beta \text{ is supported by the most reliable sensors } s_1 \text{ whose data support } \alpha$$

Essentially: combination of preferential reasoning with team-based reasoning

Motivation II: Epistemic Indistinguishability

Agents might not be able to distinguish all possible worlds

- Considered in Kripke semantics of Epistemic Modal Logic
- States are possible worlds
- Reachability relation = indistinguishability relation

Combination of with preferential reasoning in classical setting difficult

- Classical languages cannot to not have the means
- Combining preferences with Kripke-structures is cumbersome/unclear

Potential approach: combination of preferential reasoning with team-based reasoning

E.g, permits investigation of the following kind of reasoning:

$\alpha \mid\!\sim \beta$ if it is plausible that when the agent beliefs α, then also β

Motivation III: Theoretical Advancement

Preferential Reasoning has many been study with classical logics in mind

Closure under Boolean operations is not always given

- Learned system
- Also classical systems are not always Boolean, e.g., context-free languages

Team-based logics

- Connective are not classical
- Well studied and understood
- Intuition and perspectives known

Promising testbed for study preferential reasoning: combination with team-based reasoning

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Logics considered

CPL – Propositional Logic with Classical Semantics

Syntax: $\varphi ::= \Sigma \mid \neg\varphi \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$

Semantics: via valuation functions

PDL – Propositional Dependence Logic

Syntax: $\varphi ::= \Sigma \mid \neg\Sigma \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {=}\big(\vec{\Sigma}, \Sigma\big)$

Semantics: via teams (sets of valuation functions)

TPL – Propositional Logic with Team-semantics

Syntax: $\varphi ::= \Sigma \mid \neg\Sigma \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$

Semantics: via teams (sets of valuation functions)

+ their preferential versions $\mathrm{CPL}^{\mathrm{pref}}$, $\mathrm{PDL}^{\mathrm{pref}}$, and $\mathrm{TPL}^{\mathrm{pref}}$
(next section)

Propositional Dependence Logic: Syntax

Language of propositional dependence logic PDL over $\Sigma = \{p_1, \ldots, p_n\}$:

$$\varphi ::= \Sigma \mid \neg\Sigma \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid {=}\left(\vec{\Sigma}, \Sigma\right)$$

Notable aspects:

Negation only in literals

- $\neg p \vee \neg q$ (valid formula)
- $\neg(p \wedge q)$ (invalid formula)
- Not closed under negation!

Dependence atoms

- $= (a_1, \ldots, a_m, b)$
- value of b depends on $a_1, \ldots, a_m$

Example formulas:

- $\neg g \vee a$ (g „implies“ a)
- $=(a, b) \wedge a$ (a holds and b depends on a)
- $=(c)$ (c has always the same value)

Classical Interpretations (propositional logic):

$$\Omega^{\mathsf{c}} = \{\ v \mid v : \Sigma \to \{0,1\}\ \}$$

Team Semantics

Interpretations:

$$\Omega = \mathcal{P}(\Omega^{\mathsf{c}}) = \{\ X \mid X \subseteq \Omega^{\mathsf{c}}\ \}$$

For $X \in \Omega$:

$X \models p$	iff for all $v \in X$, $v \models^{\mathsf{c}} p$;
$X \models \neg p$	iff for all $v \in X$, $v \not\models^{\mathsf{c}} p$;
$X \models \bot$	iff $X = \emptyset$;
$X \models \top$	is always the case;
$X \models \alpha \wedge \beta$	iff $X \models \alpha$ and $X \models \beta$;
$X \models \alpha \vee \beta$	iff there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $Y \models \alpha$ and $Z \models \beta$;
$X \models =\!(\vec{a}, b)$	iff for all $v, v' \in X$, $v(\vec{a}) = v'(\vec{a})$ implies $v(b) = v'(b)$.

Semantics of $=\!(\vec{a}, b)$

$$X \models {=}(\vec{a}, b) \quad \text{iff} \quad \text{for all } v, v' \in X, \; v(\vec{a}) = v'(\vec{a}) \text{ implies } v(b) = v'(b).$$

Example

$X_1 =$

	a	b	c
v_1	1	1	0
v_2	1	0	1
v_3	0	1	1

$X_2 =$

	a	b	c
v_1	1	0	0
v_2	1	0	1

Evaluation works as follows:

$$X_1 \not\models {=}(a, b)$$
$$X_2 \models {=}(a, b)$$
$$X_1 \not\models {=}(b)$$

$$X_1 \models {=}(b) \vee {=}(b)$$
$$Y = \{v_1, v_3\} \qquad Z = \{v_2\}$$
$$X_1 = Y \cup Z$$
$$Y \models {=}(b) \qquad Z \models {=}(b)$$

Relational Databases

Interpret teams as database tables:

$X =$

	a	b	c
v_1	0	1	0
v_2	1	1	0
v_3	0	1	1

Formulas $\simeq$ integrity constraints.
$X \models \varphi$ amounts to checking whether φ holds for X.

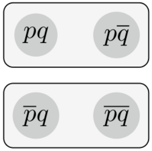
Example:
$X \models {=}(\mathtt{id}, \mathtt{col}_1) \wedge \ldots \wedge {=}(\mathtt{id}, \mathtt{col}_n)$ is a key constraint.

Inquisitive logic (The Logic of Questions[a])

Define a question operator $?p$ by:

$$X \models ?p \text{ iff } X \models p \text{ or } X \models \neg p$$

Interpret teams as answers for a question.
Answers for $?p$:



but not for $(?p) \wedge {=}(p, q)$

[a] Ciardelli: Inquisitive Logic. Springer, 2022

Entailment for Dependence Logic

Model sets:

$$\mathrm{Mod}(\alpha) = \{X \mid X \models \alpha\} \qquad \mathrm{Mod}(K) = \{X \mid \text{for all } \alpha \in K : X \models \alpha\}$$

Entailment $\models^{\mathsf{PDL}}$ and equivalence $\equiv^{\mathsf{PDL}}$ are defined as usual:

$$\alpha \models^{\mathsf{PDL}} \beta \text{ if } \mathrm{Mod}(\alpha) \subseteq \mathrm{Mod}(\beta) \qquad K \models^{\mathsf{PDL}} \beta \text{ if } \mathrm{Mod}(K) \subseteq \mathrm{Mod}(\beta)$$

$$\alpha \equiv^{\mathsf{PDL}} \beta \text{ if } \mathrm{Mod}(\alpha) = \mathrm{Mod}(\beta)$$

Intuition (databases):

$$\alpha \models^{\mathsf{PDL}} \beta \text{ if every database that complies with } \alpha \text{ also complies with } \beta$$

Intuition (sensors):

$$\alpha \models^{\mathsf{PDL}} \beta \text{ if every sensor that supports } \alpha \text{ also supports } \beta$$

Proposition

If $\alpha, \beta \in \mathrm{PL}$ are classical propositional formulas (in NNF):

$$\alpha \models^{\mathsf{TPL}} \beta \textit{ if and only if } \alpha \models^{\mathsf{PDL}} \beta \textit{ if and only if } \alpha \models^{\mathsf{CPL}} \beta$$

Possible World Interpretation – More expressible agents/knowledge bases?

Example (uncertain microbiologist):

$g \simeq$ has a certain gene
$b \simeq$ shines blue
$a \simeq$ eats amoebae

$$K = \{\, b,\, =(g,b)\,,\, \neg g \vee a \,\}$$

$=(g,b) \simeq$ blue shining is determined by having the gene
$\neg g \vee a \simeq$ ‚g implies a'

Teams that satisfy K:

$X_1 =$

	g	b	a
v_1	1	1	1
v_2	1	1	0

$X_2 =$

	g	b	a
v_3	0	1	1

Proposal for interpretation:

- Teams as model sets of K
- Two layers:
  - worlds (=assignments)
  - joint observable worlds (=teams)

We have:

$$K \not\models a$$
$$K \not\models =(a)$$
$$K \not\models g$$
$$K \models =(g)$$

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(Model-Theoretic) Logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$

- $\mathcal{L}$ Formulas
- Ω Interpretations
- $\models \subseteq \Omega \times \mathcal{L}$ Model-Relation
- $\Vdash \subseteq \mathcal{L} \times \mathcal{L}$ Entailment

Relational Model for $\mathscr{L}$

Relational Model $\langle \mathcal{S}, \ell, R \rangle$

- $\mathcal{S}$ set of states
- $\ell : \mathcal{S} \to \Omega$
- $R \subseteq \mathcal{S} \times \mathcal{S}$ binary relation on $\mathcal{S}$

' Notions:

- $\mathcal{S}(\alpha) = \{s \in \mathcal{S} \mid \ell(s) \models \alpha\}$
- $\min(\mathcal{S}(\alpha), R) = \{s \in \mathcal{S}(\alpha) \mid \nexists s' \in \mathcal{S}(\alpha) : s' R s\}$

Preferential Model for $\mathscr{L}$

Preferential Model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$

- $\langle \mathcal{S}, \ell, \prec \rangle$ is relational model
- $\prec$ strict partial order on $\mathcal{S}$
- Smoothness: for all α and $s \in S(\alpha)$
  - $s \in \min(\mathcal{S}(\alpha), \prec)$ or
  - there exist $t \prec s$ with $t \in \min(\mathcal{S}(\alpha), \prec)$

Remarks:

- $S(\alpha) \neq \emptyset$ implies $\min(\mathcal{S}(\alpha), \prec) \neq \emptyset$
- well-foundedness is not enough

Notation:

$$\min(\mathrm{Mod}(\alpha), \prec) = \{\ell(s) \mid s \in \min(\mathcal{S}(\alpha), \prec)\}$$

Preferential Entailment

Entailment $\mathrel{|\!\sim}_{\mathbb{W}}$ defined by preferential Model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$:

$$\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta \text{ if } \min(\mathcal{S}(\alpha), \prec) \subseteq \mathcal{S}(\beta) \qquad \text{(equivalently: } \min(\mathrm{Mod}(\alpha), \prec) \subseteq \mathrm{Mod}(\beta))$$

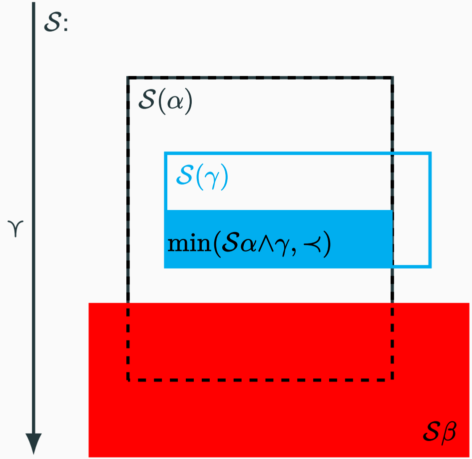
Every $\mathbb{W}$ for $\mathscr{L}$ defines a
preferential logic $\langle \mathcal{L}, \Omega, \models, \mathrel{|\!\sim}_{\mathbb{W}} \rangle$

Families of Preferential Logics

- $\mathrm{PDL}^{\mathrm{pref}}$ Preferential propositional dependence logic
- $\mathrm{CPL}^{\mathrm{pref}}$ preferential entailment of propositional logic with classical semantics
- $\mathrm{TPL}^{\mathrm{pref}}$ preferential entailment of propositional logic with team-based semantics

Illustration

Preferential Model: $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ Inference: $\alpha \mid\!\sim_{\mathbb{W}} \beta$ if $\min(\mathcal{S}(\alpha), \prec) \subseteq \mathcal{S}(\beta)$



We have the following:

- $\alpha \mid\!\sim_{\mathbb{W}} \beta$
  Because of:
  $$\min(\mathcal{S}(\alpha), \prec) \subseteq \mathcal{S}(\beta)$$
- $\alpha \wedge \gamma \not\mid\!\sim_{\mathbb{W}} \beta$
  Because of:
  $$\min(\mathcal{S}(\alpha \wedge \gamma), \prec) \not\subseteq \mathcal{S}(\beta)$$
- This is called non-monotonicity

Obligatory Bird Example

$X_F =$

	b	p	f
v_1	1	0	1

$X_P =$

	b	p	f
v_2	1	1	0

Let $\mathbb{W} = \langle \mathcal{S}_{\text{peng}}, \ell_{\text{peng}}, \prec_{\text{peng}} \rangle$ be the preferential model such that

$$\mathcal{S}_{\text{peng}} = \{s_F, s_P\} \qquad \ell_{\text{peng}}(s_i) = X_i \qquad s_F \prec_{\text{peng}} s_P$$

We obtain the following inferences:

$b \mathrel{|\!\sim}_{\mathbb{W}} f \qquad \min(\mathrm{Mod}(b), \prec_{\text{peng}}) = \{X_F\} \subseteq \mathrm{Mod}(f)$ („birds usually fly")

$p \mathrel{|\!\sim}_{\mathbb{W}} \neg f \qquad \min(\mathrm{Mod}(p), \prec_{\text{peng}}) = \{X_P\} \subseteq \mathrm{Mod}(\neg f)$ („penguins usually do not fly")

$b \wedge p \mathrel{|\!\not\sim}_{\mathbb{W}} f \qquad \min(\mathrm{Mod}(b \wedge p), \prec_{\text{peng}}) = \{X_P\} \not\subseteq \mathrm{Mod}(f)$ („penguin birds usually do not fly")

Intuition for Preferential Reasoning

CPL$^{\text{pref}}$

Typical reading:

$$\alpha \mid\!\sim_{\mathbb{W}} \beta \text{ if } \alpha\text{, then } \underline{\text{usually}}\ \beta$$

PDL$^{\text{pref}}$ and TPL$^{\text{pref}}$

Intuition (databases):

$$\alpha \mid\!\sim_{\mathbb{W}} \beta \text{ if in database that complies with } \alpha\text{, usually } \beta \text{ holds}$$

Intuition (sensors):

$$\alpha \mid\!\sim \beta \text{ if } \beta \text{ is supported by the most reliable sensors that support } \alpha$$

Inference mechanism over model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ (for logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$):

$$\alpha \mathrel{|\!\sim}_{\mathbb{W}} \beta \text{ if } \min(\mathrm{Mod}(\alpha), \prec) \subseteq \mathrm{Mod}(\beta)$$

All interpretations

Preferential Model $\mathbb{W}_{\mathsf{Full}} = \langle \mathcal{S}_{\mathsf{Full}}, \ell_{\mathsf{Full}}, \prec_{\mathsf{Full}} \rangle$

- $\mathcal{S}_{\mathsf{Full}} = \Omega$
- $\ell_{\mathsf{Full}} : \mathcal{S}_{Full} \to \Omega$, $\ell_{Full} = \mathrm{id}$
- $\prec_{\mathsf{Full}} = \emptyset$ (empty SPO)

Preferential Entailment $\mathrel{|\!\sim}_{\mathbb{W}_{Full}}$:

$$\mathrel{|\!\sim}_{\mathbb{W}_{\mathsf{Full}}} = \Vdash$$
$$\min(\mathrm{Mod}(\alpha), \prec_{\mathsf{Full}}) = \mathrm{Mod}(\alpha)$$

Empty Set of States

Preferential Model $\mathbb{W}_{\emptyset} = \langle \mathcal{S}_{\emptyset}, \ell_{\emptyset}, \prec_{\emptyset} \rangle$

- $\mathcal{S}_{\emptyset} = \emptyset$
- $\ell_{\emptyset} = \emptyset$
- $\prec_{\emptyset} = \emptyset$ (empty SPO)

Preferential Entailment $\mathrel{|\!\sim}_{\mathbb{W}_{\emptyset}}$:

$$\mathrel{|\!\sim}_{\mathbb{W}_{\emptyset}} = \mathcal{L} \times \mathcal{L}$$
$$\min(\mathrm{Mod}(\alpha), \prec_{\emptyset}) = \emptyset$$

Possible World Interpretation

Example (uncertain microbiologist):

g	$\simeq$	has a certain gene
b	$\simeq$	shines blue
a	$\simeq$	eats amoebae

$$K = \{\, b,\ =\!(g,b)\ ,\ g \vee a \,\}$$

$=\!(g,b)$	$\simeq$	blue shining is determined by having the gene
$g \vee a$	$\simeq$	‚$\neg g$ implies a‘

Teams that satisfy K:

$X_1 =$

	g	b	a
v_1	1	1	1
v_2	1	1	0

$X_2 =$

	g	b	a
v_3	0	1	1

Preferences for X_1 and X_2:

- X_1 more plausible than X_2

Preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$:

$$\mathcal{S} = \{s_1, s_2\} \qquad \ell(s_1) = X_1$$
$$s_1 \prec s_2 \qquad \ell(s_2) = X_2$$

We have:

$$K \not\models a \qquad K \not\mid\!\sim_{\mathbb{W}} a \qquad K \cup \{\neg g\} \mid\!\sim_{\mathbb{W}} a$$
$$K \not\models =\!(a) \qquad K \not\mid\!\sim_{\mathbb{W}} =\!(a) \qquad K \cup \{\neg g\} \mid\!\sim_{\mathbb{W}} =\!(a)$$
$$K \not\models g \qquad K \mid\!\sim_{\mathbb{W}} g \qquad K \cup \{a\} \not\mid\!\sim_{\mathbb{W}} g$$
$$K \models =\!(g) \qquad K \mid\!\sim_{\mathbb{W}} =\!(g) \qquad K \cup \{a\} \mid\!\sim_{\mathbb{W}} =\!(g)$$

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$$\frac{}{\alpha \mid\!\sim \alpha} \quad \text{(Ref)} \qquad \frac{\alpha \models^{\mathsf{CPL}} \beta \quad \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta} \quad \text{(RW)}$$

$$\frac{\alpha \equiv^{\mathsf{CPL}} \beta \quad \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma} \quad \text{(LLE)} \qquad \frac{\alpha \mid\!\sim \beta \quad \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma} \quad \text{(CM)}$$

$$\frac{\alpha \wedge \beta \mid\!\sim \gamma \quad \alpha \mid\!\sim \beta}{\alpha \mid\!\sim \gamma} \quad \text{(Cut)} \qquad \frac{\alpha \mid\!\sim \gamma \quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma} \quad \text{(Or)}$$

System C = (Ref) + (RW) + (LLE) + (CM) + (Cut)

System P = System C + (Or)

Proposition (Kraus, Lehmann und Magidor, 1990)

The following statements are equivalent for CPL:

- $\mid\!\sim$ *satisfies **System P***
- $\mid\!\sim$ *is preferential* *(there is a preferential model $\mathbb{W}$ with $\mid\!\sim \; = \; \mid\!\sim_{\mathbb{W}}$)*

Axiomatic Characterization For Free?

$$\frac{}{\alpha \mathrel{|\!\sim} \alpha} \quad \text{(Ref)} \qquad \frac{\alpha \models \beta \quad \gamma \mathrel{|\!\sim} \alpha}{\gamma \mathrel{|\!\sim} \beta} \quad \text{(RW)}$$

$$\frac{\alpha \equiv \beta \quad \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma} \quad \text{(LLE)} \qquad \frac{\alpha \mathrel{|\!\sim} \beta \quad \alpha \mathrel{|\!\sim} \gamma}{\alpha \land \beta \mathrel{|\!\sim} \gamma} \quad \text{(CM)}$$

$$\frac{\alpha \land \beta \mathrel{|\!\sim} \gamma \quad \alpha \mathrel{|\!\sim} \beta}{\alpha \mathrel{|\!\sim} \gamma} \quad \text{(Cut)} \qquad \frac{\alpha \mathrel{|\!\sim} \gamma \quad \beta \mathrel{|\!\sim} \gamma}{\alpha \lor \beta \mathrel{|\!\sim} \gamma} \quad \text{(Or)}$$

System C = (Ref) + (RW) + (LLE) + (CM) + (Cut)

System P = System C + (Or)

Theorem

For PDL, the following is equivalent:

- $\mathrel{|\!\sim}$ *satisfies **System P***
- $\mathrel{|\!\sim}$ *is preferential*

statement does not hold!

Proposition

For PDL*, every preferential entailment relation* $\mid\!\sim_{\mathbb{W}}$ *satisfies System C.*

Proposition

There is a preferential entailment relation $\mid\!\sim_{\mathbb{W}}$ *for* PDL *that violates* (Or)*.*

So... $\mathrm{PDL}^{\mathrm{pref}}$

- satisfies System C, but
- violates **System P**.

Violation of (Or) by PDL: Preferences are not required

$$\frac{\alpha \mid\!\sim \gamma \qquad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma} \tag{Or}$$

Setting:

$\alpha = \beta = \gamma = {=}(p)$ so, we have: $\alpha \models \gamma$ $\beta \models \gamma$

Consider the team X over $\{p, q\}$ defined by:

	p	q
v_1	1	0
v_2	0	1

We obtain

$X \not\models \alpha$ $X \not\models \beta$ $X \not\models \gamma$ $X \models \alpha \vee \beta$

Consequently, $\alpha \vee \beta \not\models \gamma$

Regain **System P** Satisfaction

Theorem

Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ *be a preferential model for* PDL.
The following statements are equivalent:

(i) $\mid\!\sim_{\mathbb{W}}$ *satisfies **System P**.*

(ii) *The* $(\star)$*-property holds:*

$$\min(\mathrm{Mod}(\alpha \vee \beta), \prec) \subseteq \min(\mathrm{Mod}(\alpha), \prec) \cup \min(\mathrm{Mod}(\beta), \prec) \qquad (\star)$$

(iii) *The* $\triangle$*-property holds:*

$$\text{for all } s \in \mathcal{S} \text{ with } |\ell(s)| > 1 \text{ exists } s' \in \mathcal{S} \text{ with } \ell(s') \subsetneq \ell(s) \text{ and } s' \prec s \qquad (\triangle)$$

Regain **System P** Satisfaction: Proof Idea (ii) $\Leftrightarrow$ (iii)

(ii) The $(\star)$-property holds:

$$\min(\mathrm{Mod}(\alpha \vee \beta), \prec) \subseteq \min(\mathrm{Mod}(\alpha), \prec) \cup \min(\mathrm{Mod}(\beta), \prec) \qquad (\star)$$

(iii) The $\triangle$-property holds:

$$\text{for all } s \in \mathcal{S} \text{ with } |\ell(s)| > 1 \text{ exists } s' \in \mathcal{S} \text{ with } \ell(s') \subsetneq \ell(s) \text{ and } s' \prec s \qquad (\triangle)$$

(ii) $\Rightarrow$ (iii):

- Show $(\square)$: for each $|X| > 1$ there is Y with:

$$Y \subsetneq X \ , \qquad Y \neq \emptyset \ , \qquad Y \prec X$$

- For $|X| \geq 1$ there is a formulas Θ_X:

$$Y \models \Theta_X \text{ iff } Y \subseteq X$$

- $\Theta_X = \bigvee_{v \in X} (p_1^v \wedge \cdots \wedge p_n^v)$

- For $|X| > 1$ there are non-empty $Y, Z \subsetneq X$:
  $X \not\models \Theta_Y$ and $X \not\models \Theta_Z$
- $\alpha = \Theta_Y$ and $\beta = \Theta_Z$ in $(\star)$:

$$\min(S(\alpha \vee \beta), \prec) \subseteq \min(S(\alpha), \prec) \cup \min(S(\beta), \prec)$$

- Hence, $X \notin \min(\mathrm{Mod}(\alpha \vee \beta), \prec)$
- Iteration of $(\square)$ yields $(\triangle)$

(iii) $\Rightarrow$ (ii):

- $\min(\mathrm{Mod}(\alpha), \prec)$ are only singleton teams
- $(\star)$ inherited from classical logic

Flattening of ϕ:

formula $\phi^f := \phi[=(\dots) \,/\, \top]$; i.e., replacing all dependence atoms by $\top$.

Example:

$$\phi = a \vee {=}(a,b) \qquad\qquad \phi^f = a \vee \top$$

Theorem

Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model over PDL *that satisfies **System P**.*
For all α, β holds:

$$\alpha \mathrel{|\!\!\sim}_{\mathbb{W}} \beta \quad \text{if and only if} \quad \alpha^f \mathrel{|\!\!\sim}_{\mathbb{W}'} \beta^f$$

where

- $\mathbb{W}' = \langle \mathcal{S}', \ell', \prec' \rangle$ *denotes a preferential model for* CPL
- *obtained from $\mathbb{W}$ by consider only singleton teams.*

Subset-Order

Preferential Model $\mathbb{W}_{\text{sub}} = \langle \mathcal{S}_{\text{sub}}, \ell_{\text{sub}}, \prec_{\text{sub}} \rangle$

- $\mathcal{S}_{\text{sub}} = \Omega \setminus \{\emptyset\}$
- $\ell_{\text{sub}} : \mathcal{S}_{\text{sub}} \to \mathcal{S}_{\text{sub}}$, $\ell_{\text{sub}} = \text{id}$
- $\prec_{\text{sub}} = \subsetneq$

Preferential Entailment $\mid\!\sim_{\mathbb{W}_{\text{sub}}}$:

$$\alpha \mid\!\sim_{\mathbb{W}_{\text{sub}}} \beta \;\text{ iff }\; \alpha^f \models^c \beta^f$$

(Theorem from last slide)

Supset-Order

Preferential Model $\mathbb{W}_{\text{sup}} = \langle \mathcal{S}_{\text{sup}}, \ell_{\text{sup}}, \prec_{\text{sup}} \rangle$

- $\mathcal{S}_{\text{sup}} = \Omega \setminus \{\emptyset\}$
- $\ell_{\text{sup}} : \mathcal{S}_{\text{sup}} \to \mathcal{S}_{\text{sup}}$, $\ell_{\text{sup}} = \text{id}$
- $\prec_{\text{sup}} = \supsetneq$

Preferential Entailment $\mid\!\sim_{\mathbb{W}_{\text{sup}}}$:

$$\mid\!\sim_{\mathbb{W}_{\text{sup}}} = \models$$

(due to downward-closure)

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Conclusion

Proposition

The following statements hold:

(a) PDL *satisfies* ***System C****, but violates* ***System P****.*

(b) TPL *satisfies* ***System P****.*

Proposition

Let $\mathbb{W}$ *be a preferential model for* TPL*.*
If $(\star)$ *is satisfied for all formulas* A, B*, then* $\mid\!\sim_{\mathbb{W}}$ *satisfies* ***System P****, whereby:*

$$\min(\mathrm{Mod}(A \vee B), \prec) \subseteq \min(\mathrm{Mod}(A), \prec) \cup \min(\mathrm{Mod}(B), \prec) \qquad (\star)$$

But, the converse is not always true.

Violation of (Or) for $\mathrm{TPL}^{\mathrm{pref}}$

Proposition

The entailment relation $\mid\!\sim_{\mathbb{W}_{pq}}$ *for* TPL *violates* (Or).

We consider the teams $X_{pq} = \{v_1\}$, $X_{\overline{p}q} = \{v_2\}$, and $X_{p\leftrightarrow q} = \{v_1, v_3\}$:

$X_{pq} =$

	p	q
v_1	1	1

$X_{\overline{p}q} =$

	p	q
v_2	0	1

$X_{p\leftrightarrow q} =$

	p	q
v_1	1	1
v_3	0	0

Let $\mathbb{W}_{\mathsf{pq}} = \langle \mathcal{S}_{\mathsf{pq}}, \ell_{\mathsf{pq}}, \prec_{\mathsf{pq}} \rangle$ be the preferential model such that

$$\mathcal{S}_{\mathsf{pq}} = \{s_{pq}, s_{\overline{p}q}, s_{p\leftrightarrow q}\} \qquad \ell_{\mathsf{pq}}(s_X) = X \qquad X_{p\leftrightarrow q} \prec_{\mathsf{pq}} X_{pq} \qquad X_{p\leftrightarrow q} \prec_{\mathsf{pq}} X_{\overline{p}q}$$

We obtain the following preferential entailments:

$$p \mid\!\sim_{\mathbb{W}_{\mathsf{pq}}} q \qquad \neg p \mid\!\sim_{\mathbb{W}_{\mathsf{pq}}} q \qquad p \vee \neg p \mid\!\not\sim_{\mathbb{W}_{\mathsf{pq}}} q \qquad \left[\ \frac{\alpha \mid\!\sim \gamma \quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}\ (\mathrm{Or})\ \right]$$

$\mathrm{TPL}^{\mathrm{pref}}$: Apparent Paradoxical Situation

- $\mathrm{CPL}^{\mathrm{pref}}$ satisfies **System P**
- $\mathrm{TPL}^{\mathrm{pref}}$ violates **System P**

Proposition

If $\alpha, \beta \in \mathrm{PL}$ are classical propositional formulas (in NNF):

$$\alpha \models^{\mathsf{TPL}} \beta \textit{ if and only if } \alpha \models^{\mathsf{CPL}} \beta$$

On one side with have flatness:

$$[\text{Flatness}] \;\; X \models \alpha \iff \text{for all } v \in X, \; \{v\} \models \alpha.$$

On the other side, preferential models over teams are more expressive.
Furthermore:

- TPL is a fragment of PDL
- $\mathrm{TPL}^{\mathrm{pref}}$ and $\mathrm{PDL}^{\mathrm{pref}}$ are different

Introduction and Motivation

1.

Logic Background

2.

Preferential Logics

Axiomatics

3.

Complexity

4.

Conclusion

System P

$$\frac{}{\alpha \mid\sim \alpha} \quad \text{(Ref)} \qquad \frac{\alpha \models^c \beta \quad \gamma \mid\sim \alpha}{\gamma \mid\sim \beta} \quad \text{(RW)}$$

$$\frac{\alpha \equiv^c \beta \quad \alpha \mid\sim \gamma}{\beta \mid\sim \gamma} \quad \text{(LLE)} \qquad \frac{\alpha \mid\sim \beta \quad \alpha \mid\sim \gamma}{\alpha \wedge \beta \mid\sim \gamma} \quad \text{(CM)}$$

$$\frac{\alpha \wedge \beta \mid\sim \gamma \quad \alpha \mid\sim \beta}{\alpha \mid\sim \gamma} \quad \text{(Cut)} \qquad \frac{\alpha \mid\sim \gamma \quad \beta \mid\sim \gamma}{\alpha \vee \beta \mid\sim \gamma} \quad \text{(Or)}$$

Conditional (assertion): $\alpha \mid\sim \beta$

Conditional Knowledge base: set of conditionals $\mathbf{K} = \{\alpha_1 \mid\sim \beta_1, \alpha_2 \mid\sim \beta_2 \ldots\}$

Entailment: $\mathbf{K} \models^{\mathsf{P}} \alpha \mid\sim \beta$ if there is a **System P** proof for $\alpha \mid\sim \beta$ when assuming $\mathbf{K}$

Proposition (Lehmann und Magidor, 1992)

The following statements are equivalent:

- $\mathbf{K} \models^{P} \alpha \mid\sim \beta$
- $\alpha \mid\sim \beta$ *holds in all preferential models that satisfy* $\mathbf{K}$ ($\mathbf{K} \subseteq \mid\sim_{\mathbb{W}}$ implies $\alpha \mid\sim_{\mathbb{W}} \beta$)

Proposition (Lehmann und Magidor, 1992)

The following problem is coNP-complete:

ENTAILMENT

Given: *Finite conditional knowledge base* $\mathbf{K}$ *and a conditional* $\alpha \mathrel{|\!\sim} \beta$

Question: *Does* $\mathbf{K} \models^{\mathsf{P}} \alpha \mathrel{|\!\sim} \beta$ *hold?*

$$\mathbf{K}^{\mathsf{SysP}} = \{\alpha \mathrel{|\!\sim} \beta \mid \mathbf{K} \models^{\mathsf{P}} \alpha \mathrel{|\!\sim} \beta\}$$

- closure operator
- $\mathbf{K}^{\mathsf{SysP}}$ satisfies **System P**

Proposition

The following statements are equivalent:

- $\mathrel{|\!\sim}$ *satisfies **System P***
- $\mathrel{|\!\sim}$ *is preferential* (*preferential model* $\mathbb{W}$ *with* $\mathrel{|\!\sim} = \mathrel{|\!\sim}_{\mathbb{W}}$)
- *there is some* $\mathbf{K}$ *with* $\mathrel{|\!\sim} = \mathbf{K}^{\mathsf{SysP}}$

$\mathsf{PDL}^{\mathsf{pref}}$: Complexity of Preferential Entailment (w.r.t. $\mathbb{W}$)

Representational Issues for $\mathbb{W}$:

- Preferential models can be huge; $\Omega(2^{2^n})$, where n is the number of atoms
- Complexity results for the general case and succinct representations[1]

$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$
Input: Formulas α, β,
preferential model $\mathbb{W}$
Question: $\alpha \mid\!\sim_{\mathbb{W}} \beta$?

Theorem

$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$ *is in* Θ_2^p *and* NP*-hard.*

$\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$
Input: Formulas α, β,
succinctly given preferential model $\mathbb{W}$
Question: $\alpha \mid\!\sim_{\mathbb{W}} \beta$?

Theorem

$\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$ *is in* Π_2^p *and* Δ_2^p*-hard.*

Remark:

- $\Theta_2^p = \mathsf{P}^{||\mathsf{NP}} =$ PTIME with polynomial many parallel NP-oracle queries
- $\Pi_2^p = \mathsf{coNP}^{\mathsf{NP}} =$ coNP with NP-oracle queries
- $\Delta_2^p =$ P with NP-oracle queries

[1] By two $(2^n)^{O(1)}$-sized circuits $(\mathcal{L}, \mathcal{O})$

$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$ is in Θ_2^p and NP-hard.

$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$
Input: Formulas α, β, preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$
Question: $\alpha \mid\!\sim_{\mathbb{W}} \beta$?

[Membership.]

- The model checking problem for PDL is NP-complete [Ebbing et all, 2012]
- Θ_2^p algorithm:
  1. In parallel, ask the NP-oracle for each team $T \in \mathcal{S}$ whether $T \models \alpha$ and $T \models \beta$.
  2. For every minimal element in the order induced graph $(\mathcal{S}, \prec)$, if the oracle answers were of the form $(1, 0)$ (that is, α was satisfied but β not) then reject.
  3. Accept.

[Hardness.]

- The model checking problem for PDL is NP-complete [Ebbing et all, 2012].
- Reduce model checking for PDL into $\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$:

$$(T, \alpha) \mapsto ((\{T\}, \mathrm{id}_{\mathcal{S}}, \emptyset), \top, \alpha).$$

$\textsc{SuccEnt}(\mathrm{PDL}^{\mathrm{pref}})$ is in Π_2^p and Δ_2^p-hard.

$\textsc{SuccEnt}(\mathrm{PDL}^{\mathrm{pref}})$
Input: Formulas α, β, succinctly given preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$
Question: $\alpha \mid\!\sim_{\mathbb{W}} \beta$?

Definition (Succinct representation $(\mathcal{L}, \mathcal{O})$)

Setting:

- N set of propositions with $|N| = n$,
- $\mathcal{S} = \{0,1\}^m$ be a set for $m \in (2^n)^{O(1)}$,
- $\prec \subseteq \mathcal{S} \times \mathcal{S}$ be a strict partial order.

For a preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ define two $(2^n)^{O(1)}$-sized circuit families $\mathcal{L}, \mathcal{O}$ (labelling, ordering) such that the following is true:

1. ℓ is computed by $\mathcal{L}$,
2. $\mathcal{O} \colon \mathcal{S} \times \mathcal{S} \rightharpoonup \{0,1\}$ is a partial function such that for $s, s' \in \mathcal{S}$, the circuit outputs 1 if and only if $s \prec s'$ is true.

[Membership.] Π_2^p algorithm:

1. Univerisally nondeterministically branch on all elements $s \in \mathcal{S}$ specified by inputs to $\mathcal{O}$, and all assignments $j \colon \{x_1, \dots, x_n\} \to \{0,1\}$.
2. Existentially nondeterministically branch on all $s' \in \mathcal{S}$
3. If $j \neq \mathcal{L}(s)$ then accept.
4. If $j \not\models \alpha$ or $j \models \beta$ then accept.
5. If $\mathcal{L}(s') \models \alpha$ and $\mathcal{O}(s', s)$ then accept.
6. Reject.

[Hardness.] Reduction from $\textsc{OddLexMaxSAT}$ which is Δ_2^p-complete [Krentel, 1988]

Complexity Results: Summary

Problem	Tract.	Complexity
$\textsc{Ent}(\mathsf{CPL}^{\mathsf{pref}})$	✓	$\in$ P, NC^1-hard
$\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})_{>_{\mathrm{lex}}}$	✗	Δ_2^p-complete
$\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard
$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$	✗	$\in \Theta_2^p$, NP-hard
$\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard
$\textsc{Ent}(\mathsf{TPL}^{\mathsf{pref}})$	✓	$\in$ P, NC^1-hard
$\textsc{SuccEnt}(\mathsf{TPL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard

Remarks:

- Complexity of $\textsc{Ent}$ and $\textsc{SuccEnt}$ for $\mathsf{CPL}^{\mathsf{pref}}$ was **not** known before
- Tight results are surprisingly hard to proof

Introduction and Motivation

1. Logic Background

2. Preferential Logics

3. Axiomatics

4. Complexity

Conclusion

Summary

Content

- Motivation
  - Sensor data, epistemic indistinguishability, ...
- Team-Semantics (PDL,TPL)
- Preferential Logics ($\mathrm{PDL}^{\mathrm{pref}}$,$\mathrm{TPL}^{\mathrm{pref}}$,$\mathrm{CPL}^{\mathrm{pref}}$)
- Axiomatics of $\mathrm{PDL}^{\mathrm{pref}}$ (and $\mathrm{TPL}^{\mathrm{pref}}$)
  - Cumulative (**System C**)
  - Violation of **System P**
  - **System P** Characterization
  - $\mathrm{TPL}^{\mathrm{pref}}$ is not fragment of $\mathrm{PDL}^{\mathrm{pref}}$
- Complexity
  - $\textsc{Ent}$ and $\textsc{SuccEnt}$
  - (In-)Tractability

Problem	Tract.	Complexity
$\textsc{Ent}(\mathrm{CPL}^{\mathrm{pref}})$	✓	$\in$ P, NC^1-hard
$\textsc{SuccEnt}(\mathrm{CPL}^{\mathrm{pref}})_{>_{\mathrm{lex}}}$	✗	Δ_2^p-complete
$\textsc{SuccEnt}(\mathrm{CPL}^{\mathrm{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard
$\textsc{Ent}(\mathrm{PDL}^{\mathrm{pref}})$	✗	$\in \Theta_2^p$, NP-hard
$\textsc{SuccEnt}(\mathrm{PDL}^{\mathrm{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard
$\textsc{Ent}(\mathrm{TPL}^{\mathrm{pref}})$	✓	$\in$ P, NC^1-hard
$\textsc{SuccEnt}(\mathrm{TPL}^{\mathrm{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard

Future work

- Complexity results (tight)
- More on axiomatizations
- Other team-based logics
- Conditionals

Thanks for your attention!