Propositional Variable Forgetting and Marginalization: Semantically, Two Sides of the Same Coin

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Abstract. This paper investigates variable forgetting and marginalization in propositional logic. We show that for finite signatures and infinite signatures, variable forgetting and marginalization are corresponding operations, i.e., they yield semantically equivalent outputs for respective complementary inputs. This observation holds for formulas and also for sets of formulas. For formulas, both operations, variable forgetting and marginalization, are shown to be compatible with disjunctions, but not with conjunction, implication and negation. For general sets of formulas, a consequence is that the element-wise application of these operations to a set of formulas and the application to a formula equivalent to this set are not equivalent in general. However, for every deductively closed set X, we show that the element-wise application of variable forgetting or marginalization, respectively, and the application to any formula equivalent to X are equivalent. This latter observation is important because deductively closed sets play an important role in many areas, e.g., in logic-based approaches to knowledge representation and databases.

1 Introduction

Focusing on relevant information is a key ability of intelligent agents to reason and thus an important concern of artificial intelligence, in particular in the field of knowledge representation (KR) where research deals with representing knowledge and belief most adequately in a formal way. Whenever we set up a toy example or a large model for an application in KR, we expect that all variables that we deem to be irrelevant to our model and thus are left out, do not have any influence on reasoning processes and outcomes. For example, when we write an answer set program to provide medical knowledge for finding a best cancer therapy for a patient [25], we would like to safely *forget* variables that speak about, e.g., the weather. More precisely, we expect that the recommended therapy would be the same even if we had taken those variables into account. Similarly, in the well-known Tweety example, we expect the penguin Tweety not to fly even if we had taken dozens of other animal species into account. Such expectations might be justified by a principle analogous to what is known as the *principle of irrelevant alternatives* in social choice theory [22] – adding irrelevant variables should not change the inferences. This is also the basic idea of syntax splitting in nonmonotonic inductive reasoning and belief revision [19,13,10].

For logic-based artificial intelligence, many such forgetting approaches are known, for an overview see [6]. Several approaches considered here fall into the tradition of *variable forgetting* [17], i.e., approaches that remove signature elements from a given logical representation. Variable forgetting is also known as *variable elimination* [15] and has been studied for many formalisms like propositional logic [15], first-order logic [17], description logics [14,18], modal logic [27,24,5], etc. [8,26,16]. Closely related to variable forgetting is also the computation of the uniform interpolant [18].

Technically, in the most basic case where a variable a is to be forgotten from a propositional formula A, *variable forgetting* is implemented by the disjunction of the two modifications of A that arise if a is set to $\top$ resp. $\bot$ [15]. The resulting formula in which a does not occur anymore thus takes both possibilities of a being true or false into account, but abstracts from the specific outcome.

In probability theory, there is a similar (semantic) operation called *marginalization* [20]: the marginal probability of a formula A defined over a signature $\Sigma\backslash\{a\}$ is the sum of the probabilities of all worlds ω whose $\Sigma\backslash\{a\}$-part is a model of A and whose a-part can be positive or negative. This means that semantic marginalization executes the modifications regarding the truth values of a on the models. Marginalization can be defined in a straightforward way also for Spohn's ranking functions [23] and for total preorders [13] that both play a major role for nonmonotonic reasoning and belief revision. The question arises whether there are formal relationships between these two operations of forgetting variables in the syntax resp. semantics, and how exactly they can be made explicit. In particular, in the context of nonmonotonic reasoning resp. belief revision and syntax splitting [13,10], a most relevant question would be if the belief set induced by a marginalized ranking function or a total preorder, respectively, coincides with the result of applying the syntactic operation of variable forgetting to the belief set of the original function. Since belief sets are deductively closed, this leads us naturally to investigations of variable forgetting on sets of formulas, its interactions with Boolean connectives, and its behaviour under deductive closure.

In this paper, we elaborate on the correspondences between variable forgetting and marginalization in propositional logic for finite and infinite signatures. Our main contributions are summarized in the following[3]:

(1) *Variable forgetting and marginalization are two sides of the same coin, semantically.* We show that for a formula φ over Σ, the marginalization of $Mod(\varphi)$ to Γ yields the same as the models of forgetting variables $\Sigma \setminus \Gamma$ in φ. This give reason to define the novel notion of *syntactic marginalization of* Γ, defined as forgetting variables $\Sigma \setminus \Gamma$ in φ. Our results carry over to sets.

[3] The proofs for the propositions and theorems of this paper are available in the accompanying supplementary material.

(2) *Determine the compatibility of syntactic marginalization with Boolean connectives.* We consider whether marginalization of a complex formula is equivalent to performing marginalization of the sub-formulas instead. Our investigations show that syntactic marginalization, and thus also variable forgetting, are compatible with disjunction, but incompatible with conjunction, negation and implication.
(3) *Syntactic marginalizing of a set of formulas **does not** commute with marginalization of a formula that is equivalent to that set.* In general, performing syntactic marginalizing on each element of a set of formulas Γ does not yield a result equivalent to the syntactic marginalization of φ with $\varphi \equiv \Gamma$.
(4) *Syntactic marginalizing of a deductively closed set of formulas **does** commute with marginalization of a formula that is equivalent to that set.* Syntactic marginalizing of each element from a deductively closed set of formulas Γ does yield a result equivalent to the syntactic marginalization of φ with $\varphi \equiv \Gamma$.

We also show that syntactic marginalization is related to the minimal set of syntax elements required for representing a formula. Note that (3) and (4) provide insights on the marginalization-compatibility of different representations for sets of formulas. Because of (3), for knowledge-based systems that execute marginalizations, representation does matter, and one has to be careful when invoking classical equivalences between representations. Furthermore, (4) guarantees that when representing a deductive closed set of formulas K, typically also called a belief set, by an equivalent formula A, i.e., $A \equiv K$, then marginalization can be safely applied.

Note that for the formulation of properties and theorems we establish in this paper we focus on viewpoint of syntactic marginalization. Due to (1), all these properties, theorems and the contributions mentioned above also apply to variable forgetting and (model)-marginalization.

This paper is organized as follows. The next section provides the background on logic and further preliminaries. In Section 3, we establish that, semantically, variable forgetting and (model)-marginalization are dual operations. Furthermore, we define syntactic marginalization. Section 4 considers the connection between syntactic marginalization of a formula A and the minimal set of signature elements required for a formula to be equivalent to A. In Section 5, we determine the compatibility of connectives with syntactic marginalization. The general case of marginalization of arbitrary sets of formulas is considered in Section 6. The case of deductively closed sets is considered in Section 7. Section 8 discusses representational aspects and further results on the marginalization of deductively closed sets of formulas. In Section 9, we conclude and point out future work.

2 Preliminaries and Background

Let $\Sigma = \{a, b, c, \ldots\}$ be a (possible infinite) signature, whose elements are called atoms or variables, and let $\mathcal{L}_\Sigma = \{A, B, C, \ldots\}$ denote the finitely generated propositional language over Σ. For conciseness of notation, we sometimes omit the logical *and*-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e., $\overline{A}$ means $\neg A$. Furthermore, we require $\mathcal{L}$ to contain $\top$ and $\bot$, where $\top$ is interpreted, as usually, as a tautology, and $\bot$ as a contradiction. Let Ω_Σ denote the set of all possible worlds (propositional interpretations) over Σ. As usual, $\omega \models_\Sigma A$ means that the propositional formula $A \in \mathcal{L}_\Sigma$ holds in the possible world $\omega \in \Omega_\Sigma$, and $Mod_\Sigma(A) = \{\omega \mid \omega \models_\Sigma A\}$ denotes the set of all such possible worlds. With $A \equiv_\Sigma B$ we denote semantic equivalence defined as usually, i.e. $Mod_\Sigma(A) = Mod_\Sigma(B)$. With $Cn_\Sigma(A) = \{F \in \mathcal{L}_\Sigma \mid A \models_\Sigma F\}$ we denote the set of all logical consequences of A and say that $L \subseteq \mathcal{L}_\Sigma$ is *deductively closed* if $Cn_\Sigma(L) = L$. To simplify notation in the following, if Σ is finite, we will use ω both for the model and the corresponding complete conjunction containing all atoms either in positive or negative form. We say a formula A is *contingent* if $A \not\equiv \top$ and $A \not\equiv \bot$. For a set $X \subseteq \mathcal{L}_\Sigma$ of formulas, we lift the models relation to X by defining $\omega \models_\Sigma X$ if $\omega \models_\Sigma A$ for all $A \in X$. The above-mentioned notions carry over to sets of formulas in the usual way, e.g., the set of models of X is $Mod_\Sigma(X) = \{\omega \mid \omega \models_\Sigma X\}$, semantical equivalence of $A \in \mathcal{L}_\Sigma$ and X is $X \equiv_\Sigma A$ if $Mod_\Sigma(A) = Mod_\Sigma(X)$, and so forth. We assume that signatures are always non-empty sets, which applies especially to subsignatures $\Gamma \subseteq \Sigma$. If $A \in \mathcal{L}_\Sigma$ is a formula, then $\mathrm{Sig}(A)$ denotes the atoms that appear in A. Furthermore, the minimal set of signature elements (in terms of set inclusion) of a formula that is equivalent to A is denoted by $\mathrm{Sig}_{\min}(A)$. Parikh showed that $\mathrm{Sig}_{\min}(A)$ is unique and hence well-defined [19, Lem. 2]. Note that $\mathrm{Sig}_{\min}(A) = \varnothing$ holds if and only if $A \equiv \bot$ or $A \equiv \top$ holds. Moreover, negation is a neutral operation regarding minimal signatures, i.e., $\mathrm{Sig}_{\min}(A) = \mathrm{Sig}_{\min}(\neg A)$. For a deductively closed set of formulas $Cn_\Sigma(X)$ we denote its signature with $\mathrm{Sig}(Cn_\Sigma(X)) = \Sigma$.

3 Marginalization and Variable Forgetting for Formulas

In this section, we start by introducing the basic notions of variable forgetting and marginalization. Then, we will define a syntactic version of marginalization and establish the connection between these three operations.

Variable Forgetting. Forgetting in a logical setting is sometimes understood as removing a variable by an syntactic operation. The approach is rooted in the work of Lin and Reiter [17], which established a whole line of research on that topic, e.g., [15,4]. However, the technical notion but was introduced by Boole [3].

Definition 3.1 (variable forgetting [15]). *Let $A \in \mathcal{L}_\Sigma$ be a formula, let $a \in \Sigma$ be an atom, and let $\Gamma \subseteq \Sigma$ be a subsignature. The* variable forgetting of a in A*,*

$$\mathrm{VarForget}(A, a) = A[a/\top] \vee A[a/\bot] \ ,$$

arises from A by replacing all occurrences of a by $\top$, yielding $A[a/\top]$, and by replacing all occurrences of a by $\bot$, yielding $A[a/\bot]$. The variable forgetting of Γ in A*, denoted by* $\mathrm{VarForget}(A, \Gamma)$*, is the result of successively eliminating all variables of Γ in A that appear in* $\mathrm{Sig}(A)$*.*

Note that $\mathrm{VarForget}(A, \Gamma)$ is a proper propositional formula; as $\mathrm{Sig}(A)$ is always finite, also $\mathrm{VarForget}(A, \Gamma)$ is finite. It has been shown that $\mathrm{VarForget}(A, \Gamma)$

yields a syntactically equivalent formula (up to associativity of disjunction) for every order on Γ [15]. Clearly, when one is interested in the exact syntactic result of $\mathrm{VarForget}(A, \Gamma)$, order of execution matters. However, as we consider here semantic equivalences, we not investigate this further.

Example 3.2. Let $\Sigma = \{t, b, c\}$ be a signature where the atoms have following intended meaning: t stands for "tea is served", and b stands for "biscuits are served", and c stands for "coffee is served". We consider the formulas $A = (t \vee c) \to b$ ("When tea or coffee are served, then biscuits are served") and $B = (t \vee c) \wedge b$ ("tea or coffee are served, and also biscuits are served"). Various different variable eliminations in A and B are:

$$\begin{aligned}
\mathrm{VarForget}(A, t) &= ((\top \vee c) \to b) \vee ((\bot \vee c) \to b) \equiv_{\{c,b\}} c \to b \\
\mathrm{VarForget}(A, c) &= ((t \vee \top) \to b) \vee ((t \vee \bot) \to b) \equiv_{\{t,b\}} t \to b \\
\mathrm{VarForget}(A, b) &= ((t \vee c) \to \top) \vee ((t \vee c) \to \bot) \equiv_{\{t,c\}} \top \\
\mathrm{VarForget}(B, t) &= ((\top \vee c) \wedge b) \vee ((\bot \vee c) \wedge b) \equiv_{\{c,b\}} b \\
\mathrm{VarForget}(B, c) &= ((t \vee \top) \wedge b) \vee ((t \vee \bot) \wedge b) \equiv_{\{t,b\}} b \\
\mathrm{VarForget}(B, b) &= ((t \vee c) \wedge \top) \vee ((t \vee c) \wedge \bot) \equiv_{\{t,c\}} t \vee c
\end{aligned}$$

Variable forgetting of the signature $\Gamma = \{t, c\}$ in A and B yields the following:

$$\begin{aligned}
&\mathrm{VarForget}(A, \Gamma) = \mathrm{VarForget}(A, t)[c/\top] \vee \mathrm{VarForget}(A, t)[c/\bot] \\
&= (A[t/\top] \vee A[t/\bot])[c/\top] \vee (A[t/\top] \vee A[t/\bot])[c/\bot] \\
&= (((\top \vee \top) \to b) \vee ((\bot \vee \top) \to b)) \vee (((\top \vee \bot) \to b) \vee ((\bot \vee \bot) \to b)) \equiv_{\Gamma} \top \\
&\mathrm{VarForget}(B, \Gamma) = \mathrm{VarForget}(B, t)[c/\top] \vee \mathrm{VarForget}(B, t)[t/\bot] \\
&= (B[t/\top] \vee B[t/\bot])[c/\top] \vee (B[t/\top] \vee B[t/\bot])[c/\bot] \\
&= (((\top \vee \top) \wedge b) \vee ((\bot \vee \top) \wedge b)) \vee (((\top \vee \bot) \wedge b) \vee ((\bot \vee \bot) \wedge b)) \equiv_{\Gamma} b
\end{aligned}$$

A and B could be viewed as general knowledge about different serving practices. However, when being in a specific context, e.g., in a tearoom, it's sufficient (or even rational) to reason and discuss only tea and biscuits as in $\mathrm{VarForget}(A, \Gamma)$ and $\mathrm{VarForget}(B, \Gamma)$ because there will be no coffee at all.

Model Marginalization. Another approach to forgetting is marginalization, which is rooted in probability theory [20]. In contrast to variable forgetting, marginalization is defined on interpretations, using the idea of Γ-parts of interpretations. For a subsignature $\Gamma \subseteq \Sigma$ and an interpretation $\omega \in \Omega_\Sigma$ we denote the *Γ-part of* ω with $\omega^\Gamma \in \Omega_\Gamma$, mentioning exactly the atoms from Γ, i.e., $\omega^\Gamma : \Gamma \to \{0, 1\}$ with $\omega^\Gamma(a) = \omega(a)$ for all $a \in \Gamma$. Marginalization is then the reduction to the Γ-part of an interpretation.

Definition 3.3 (model marginalization, $\mathrm{ModMg}_\Sigma(\omega, \Gamma)$**,** $\mathrm{ModMg}_\Sigma(M, \Gamma)$**).** *Let $\omega \in \Omega_\Sigma$, let $M \subseteq \Omega_\Sigma$, and let $\Gamma \subseteq \Sigma$. We say* $\mathrm{ModMg}_\Sigma(\omega, \Gamma) = \omega^\Gamma$ *is the* (model) marginalization of ω from Σ to Γ. *The element-wise marginalization of all $\omega \in M$ from Σ to Γ is called* (model) marginalization of M from Σ to Γ, *denoted by* $\mathrm{ModMg}_\Sigma(M, \Gamma) = \{\mathrm{ModMg}_\Sigma(\omega, \Gamma) \mid \omega \in M\}$.

When viewing a logic as an institution [7], the marginalization of models to a subsignature as given in Definition 3.3 is just a special case of the general forgetful functor $Mod(\varphi)$ from Σ-models to Γ-models induced by any signature morphism φ from Γ to Σ. The special case of Definition 3.3 is given by the forgetful functor $Mod(\iota)$ induced by the signature inclusion $\iota : \Gamma \to \Sigma$ (cf. also [1]). In the following, we consider an example on model marginalization:

Example 3.4. Consider again $\Sigma = \{t, b, c\}$ from Example 3.2. For illustration of the model marginalization of individual interpretations, consider $\omega_1 = \overline{t}cb$ ("coffee and biscuits are served, but no tea") and $\omega_2 = \overline{t}\overline{c}b$ ("coffee is served, but no tea and biscuits") and the subsignature $\Gamma = \{t, b\}$. The Γ-part of ω_1 is $\omega_1^\Gamma = \overline{t}b$. Likewise, the Γ-part of ω_2 is the same, i.e., $\omega_2^\Gamma = \overline{t}b$. Consequently, the marginalization of these interpretations is as follows ("biscuits are served, but no tea"):

$$\mathrm{ModMg}_\Sigma(\omega_1, \Gamma) = \omega_1^\Gamma \; \mathrm{ModMg}_\Sigma(\omega_2, \Gamma) = \omega_2^\Gamma = \overline{t}b$$

The models $A = (t \vee c) \to b$ and $B = (t \vee c) \wedge b$ from Example 3.2 are:

$$Mod_\Sigma(A) = \{\, tcb,\ t\overline{c}b,\ \overline{t}cb,\ \overline{t}\overline{c}b,\ \overline{t}\overline{c}\overline{b}\,\} \qquad Mod_\Sigma(B) = \{\, tcb,\ t\overline{c}b,\ \overline{t}cb\,\}$$

Several model marginalizations of $Mod_\Sigma(A)$ and $Mod_\Sigma(B)$ are:

$$\begin{aligned}
\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \{c, b\}) &= \{\, cb,\ \overline{c}b,\ \overline{c}\overline{b}\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \{t, b\}) &= \{\, tb,\ \overline{t}b,\ \overline{t}\overline{b}\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \{t, c\}) &= \{\, tc,\ t\overline{c},\ \overline{t}c,\ \overline{t}\overline{c}\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \{b\}) &= \{\, b,\ \overline{b}\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \{c, b\}) &= \{\, cb,\ \overline{c}b\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \{t, b\}) &= \{\, tb,\ \overline{t}b\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \{t, c\}) &= \{\, tc,\ t\overline{c},\ \overline{t}c\,\} \\
\mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \{b\}) &= \{\, b\,\}
\end{aligned}$$

One can observe easily that $\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \{b\})$ is the same as the set of models of $\Sigma \setminus \mathrm{VarForget}(A, b)$, and analogously for $\mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \{b\})$ and $\Sigma \setminus \mathrm{VarForget}(B, b)$ (cf. Example 3.2). We will see that this is no coincidence, as both operations are two sides of the same coin.

Syntactic Marginalization. Note that variable forgetting is a syntactic operation on formulas and model marginalization is a semantic operation on interpretations. Moreover, variable forgetting takes the signature elements to be removed as a parameter, while for marginalization the posterior sub-signature is a parameter. To avoid this duality, we define syntactic marginalization as the dual of variable forgetting.

Definition 3.5 (syntactic marginalization). *Let $A \in \mathcal{L}_\Sigma$ and let $\Gamma \subseteq \Sigma$. The* syntactic marginalization of A (from Σ) to Γ*, written* $\mathrm{SynMg}_\Sigma(A, \Gamma)$*, is* $\mathrm{VarForget}(A, \Sigma \setminus \Gamma)$*.*

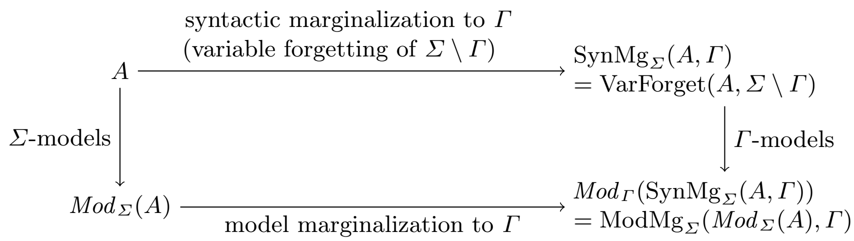

Fig. 1: Semantic compatibility between marginalization and variable forgetting.

The syntactic marginalization of a formula to a reduced signature is equivalent to the formula obtained by forgetting all variables that are not in the subsignature[4].

Interrelation. We obtain the following compatibility result for variable forgetting and marginalization which is closely related to known results on variable forgetting [15]. Figure 1 provides an illustration of these interrelations.

Theorem 3.6. *For every $A \in \mathcal{L}_{\Sigma}$ and $\Gamma \subseteq \Sigma$ the following holds:*

$$\begin{aligned} \mathrm{ModMg}_{\Sigma}(Mod_{\Sigma}(A), \Gamma) &= Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) \\ &= Mod_{\Gamma}(\mathrm{VarForget}(A, \Sigma \setminus \Gamma)) \end{aligned}$$

As a first consequence of Theorem 3.6, we obtain that despite syntactic marginalization being a pure syntactic operation, syntactic marginalization yields semantic equivalent results for semantic equivalent formulas and complies with entailment.

Corollary 3.7. *Let $A, B \in \mathcal{L}_{\Sigma}$ and let $\Gamma \subseteq \Sigma$. The following statements hold:*

(a) If $A \equiv_{\Sigma} B$, then we have $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Gamma} \mathrm{SynMg}_{\Sigma}(B, \Gamma)$.
(b) If $A \models_{\Sigma} B$, then we have $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \models_{\Gamma} \mathrm{SynMg}_{\Sigma}(B, \Gamma)$.

Because (model and syntactic) marginalization and variable forgetting comply with each other semantically, in the following sections we continue to present results from the viewpoint of syntactic marginalization. Cleary, due to Theorem 3.6 these results also carry over to variable forgetting and model marginalization.

4 Marginalization and Minimal Sets of Atoms

Before investigating syntactic marginalization in more detail, we show that $\mathrm{Sig}_{\mathrm{min}}(A)$ is the set of those atoms that distinguish models of A from non-models of A by exactly one signature element.

Proposition 4.1. *For each propositional formula $A \in \mathcal{L}_{\Sigma}$ we have:*

$$\mathrm{Sig}_{\mathrm{min}}(A) = \{a \in \Sigma \mid \exists \omega_1, \omega_2 \in \Omega .\ \omega_1 \models_{\Sigma} A \text{ and } \omega_2 \not\models_{\Sigma} A \text{ and } \omega_1^{\Sigma \setminus \{a\}} = \omega_2^{\Sigma \setminus \{a\}}\}$$

[4] We thank the anonymous reviewer for phrasing this interrelation so nicely.

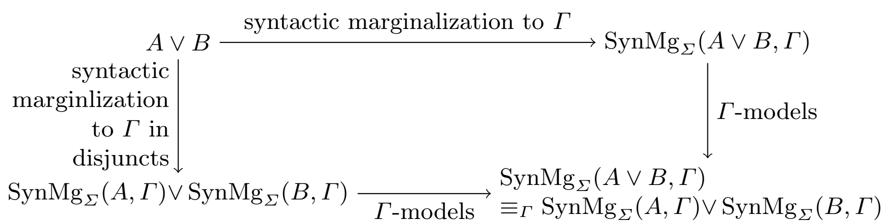

Fig. 2: Semantic compatibility between disjunction and syntactic marginalization.

Proposition 4.1 formally underpins the intuition that a minimal formula (in terms of different atoms) is required to make use of exactly those atoms which can be distinguished semantically.

From Theorem 3.6 and Proposition 4.1, we obtain the following connection between syntactic marginalization and $\mathrm{Sig}_{\min}(A)$.

Proposition 4.2. *Let Σ and Γ be signatures with $\Gamma \subseteq \Sigma$ and let $A \in \mathcal{L}_{\Sigma}$. The following statements hold:*

(a) We have that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) \subseteq \mathrm{Sig}_{\min}(A) \cap \Gamma$ holds.
(b) We have that $a \in \mathrm{Sig}_{\min}(A)$ if and only if $A \not\equiv_{\Sigma} \mathrm{SynMg}_{\Sigma}(A, \Sigma \setminus \{a\})$ holds.
(c) If A is consistent, then $\Gamma \cap \mathrm{Sig}_{\min}(A) = \varnothing$ if and only if $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Sigma} \top$.

5 Compatibility of Syntactic Marginalization with Connectives

We now investigate the compatibility of syntactic marginalization with the standard connectives of propositional logic. The compatibility with disjunction and conjunction over finite signatures was investigated for propositional logic from the perspective for variable forgetting by Zhang and Zhou [27]. We show that syntactic marginalization is fully compatible with disjunctions and only partially compatible with the $\wedge$ connective and also with the $\neg$ connective of propositional logic both for finite and infinite signatures. We start with the compatibility of syntactic marginalization with disjunction.

Proposition 5.1. *For each $A \equiv_{\Sigma} A_1 \vee \ldots \vee A_n$ with $A, A_1, \ldots A_n \in \mathcal{L}_{\Sigma}$ and each $\Gamma \subseteq \Sigma$ the following holds:*

$$\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Gamma} \mathrm{SynMg}_{\Sigma}(A_1, \Gamma) \vee \ldots \vee \mathrm{SynMg}_{\Sigma}(A_n, \Gamma)$$

For conjunction we consider both directions of semantic equivalence separately. In general syntactic marginalization and conjuncts are not compatible; however, the following proposition shows that one direction of semantic equivalence holds.

Proposition 5.2. *For each $A, A_1, \ldots A_n \in \mathcal{L}_{\Sigma}$ with $A \equiv_{\Sigma} A_1 \wedge \ldots \wedge A_n$ and for each $\Gamma \subseteq \Sigma$ the following holds:*

$$\mathit{Mod}_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) \subseteq \mathit{Mod}_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A_1, \Gamma) \wedge \ldots \wedge \mathrm{SynMg}_{\Sigma}(A_n, \Gamma)) \quad (1)$$

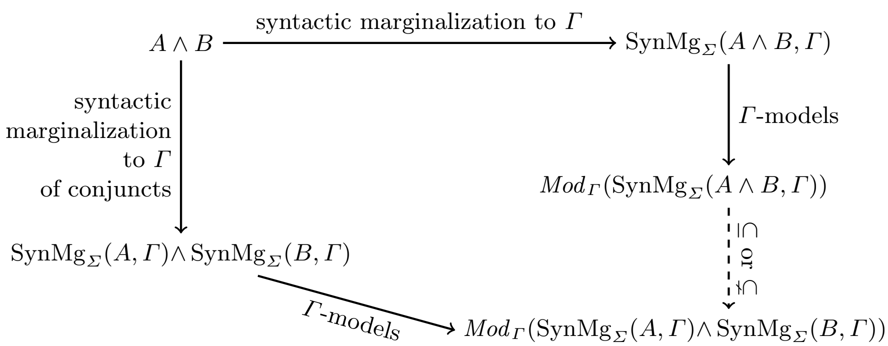

Fig. 3: Semantic relations between conjunction and syntactic marginalization with respect to Γ. The relation between Γ-models of $\mathrm{SynMg}_\Sigma(A \wedge B, \Gamma)$ and of $\mathrm{SynMg}_\Sigma(A, \Gamma) \wedge \mathrm{SynMg}_\Sigma(B, \Gamma)$ is a subset-relation (represented by the dashed arrow); in certain cases this subset-relation is strict.

The converse direction of Equation (3) in Proposition 5.2 does not hold in general. In the following proposition, we present this claim formally.

Proposition 5.3. *Let Σ be a signature with three or more elements. There exist $A, A_1, \ldots A_n \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma A_1 \wedge \ldots \wedge A_n$ and $\Gamma \subseteq \Sigma$ such that:*

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \ldots \wedge \mathrm{SynMg}_\Sigma(A_n, \Gamma)) \not\subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))$$

Figure 3 illustrates and summarizes our observations on the compatibility between conjunction and syntactic marginalization. Next, we consider an example on Proposition 5.3.

Example 5.4. Suppose that $\Sigma = \{a, s, f\}$ is a signature, where a has the intended meaning "is an animal", and s stands for "can swim", and f stands for "has fins". We consider the formulas $A = a \wedge s \wedge f$ ("It is an animal that can swim and has fins."), $A_1 = a \wedge (s \leftrightarrow f)$ ("It is an animal and it can swim if and only if it has fins.") and $A_2 = a \wedge f$ ("It is an animal with fins."). The syntactic marginalizations of these formulas to $\Gamma = \{a, s\}$ are:

$$\mathrm{SynMg}_\Sigma(A, \Gamma) = (a \wedge s \wedge \top) \vee (a \wedge s \wedge \bot) \equiv a \wedge s \quad \text{("It is an animal that can swim.")}$$

$$\mathrm{SynMg}_\Sigma(A_1, \Gamma) = (a \wedge (s \leftrightarrow \top)) \vee (a \wedge (s \leftrightarrow \bot)) \equiv a \quad \text{("It is an animal.")}$$

$$\mathrm{SynMg}_\Sigma(A_2, \Gamma) = (a \wedge \top) \vee (a \wedge \bot) \equiv a \quad \text{("It is an animal.")}$$

We observe that $A \equiv A_1 \wedge A_2$ holds, yet $\mathrm{SynMg}_\Sigma(A, \Gamma)$ differs semantically from $\mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \mathrm{SynMg}_\Sigma(A_2, \Gamma)$. More intuitively speaking, the information about s, which is clearly stated in A, has got lost in the forgetting of f from $A_1 \wedge A_2$. This is because the conjunction $A_1 \wedge A_2$ encodes the truth of s via a dependence of s from f, which is forgotten.

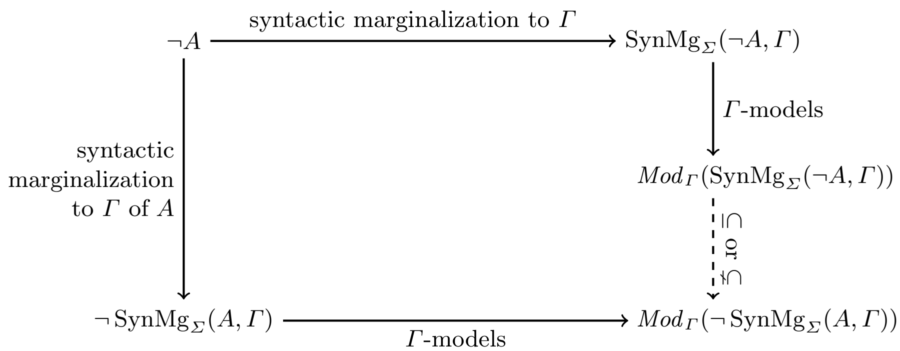

Fig. 4: Semantic relations between negation and syntactic marginalization with respect to Γ. The relation between Γ-models of $\mathrm{SynMg}_\Sigma(A, \Gamma)$ and of $\neg\, \mathrm{SynMg}_\Sigma(A, \Gamma)$ is a subset-relation (represented by the dashed arrow); in certain cases this subset-relation is strict.

For the case of negation we obtain results analogue to the case of conjunction. Syntactic marginalization is in general not fully compatible with negation; however, the following proposition attests that one direction of semantic equivalence holds.

Proposition 5.5. *For all $A, B \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma \neg B$ and for all $\Gamma \subseteq \Sigma$ we have:*

$$Mod_\Gamma(\neg(\mathrm{SynMg}_\Sigma(B, \Gamma))) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))$$

The next proposition states that the inclusion (9) in Proposition 5.5 is sometimes strict. In contrast to conjunction, the incompatibility arizes already for signatures of size two.

Proposition 5.6. *Let Σ be a signature with two or more elements and let $\Gamma \subsetneq \Sigma$ be a strict subsignature. There are formulas $A, B \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma \neg B$ such that:*

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma)) \not\subseteq Mod_\Gamma(\neg\, \mathrm{SynMg}_\Sigma(B, \Gamma))$$

Figure 4 summarizes the results presented here on the compatibility of marginalization and Boolean negation.

Example 5.7. Suppose that $\Sigma = \{d, s, c\}$ is a signature, where d has the intended meaning "is a doctor", and s stands for "wears a stethoscope", and c stands for "wears a doctor's coat". We consider the formulas $A = d \to (s \lor c)$ ("A doctor wears a stethoscope or a coat.") and $B = d \land \neg s \land \neg c$ ("A doctor without a stethoscope who wears no coat."). The syntactic marginalizations to $\Gamma = \{d, s\}$ of these formulas are:

$$\mathrm{SynMg}_\Sigma(A, \Gamma) = (d \to (s \lor \top)) \lor (d \to (s \lor \bot)) \qquad \equiv \top \qquad \text{(“Tautology.”)}$$
$$\mathrm{SynMg}_\Sigma(B, \Gamma) = (d \land \neg s \land \top) \lor (d \land \neg s \land \bot) \qquad \equiv d \land \neg s$$

("A doctor without a stethoscope.")

Clearly, we can observe that $A \equiv \neg B$ holds, yet $\mathrm{SynMg}_{\Sigma}(A, \Gamma)$ differs semantically from $\neg\, \mathrm{SynMg}_{\Sigma}(B, \Gamma)$.

Recall that minimal signatures are invariant under negation, i.e., we have $\mathrm{Sig}_{\min}(A) = \mathrm{Sig}_{\min}(\neg A)$ for each formula A. A consequence of this and Proposition 5.6 (which is also witnessed by Example 5.7) is that the result of syntactic marginalization depends on semantic content, and not just on the atoms required for representation.

Corollary 5.8. *For each signature Σ with two or more elements, there exist formulas $A, B \in \mathcal{L}_{\Sigma}$ with $\mathrm{Sig}_{\min}(A) = \mathrm{Sig}_{\min}(B)$ such that $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \neq \mathrm{SynMg}_{\Sigma}(B, \Gamma)$.*

Another consequence of the results given above is the following result on syntactic marginalization and implication.

Proposition 5.9. *For each $A, B, C \in \mathcal{L}_{\Sigma}$ with $A \equiv_{\Sigma} B \to C$ and for each $\Gamma \subseteq \Sigma$ the following holds:*

$$\mathit{Mod}_{\Gamma}(\mathrm{SynMg}_{\Sigma}(B, \Gamma) \to \mathrm{SynMg}_{\Sigma}(C, \Gamma)) \subseteq \mathit{Mod}_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma))$$

However, as in the case of conjunction and negation, syntactic marginalization does not comply with implication.

Example 5.10. Let $\Sigma = \{a, b, \ldots\}$ and let $\Gamma \subseteq \Sigma$ be a subsignature such that $a \notin \Gamma$ and $b \in \Gamma$. We choose the formulas $A = a \wedge b$ and $B = \neg a \vee \neg b$ and $C = \bot$. One can observe easily that $A \equiv_{\Sigma} B \to C = \neg B \vee C$ holds.

$$\begin{aligned}
&\mathrm{SynMg}_{\Sigma}(A, \Gamma) = (\top \wedge b) \vee (\bot \wedge b) \equiv_{\Gamma} b \\
&\mathrm{SynMg}_{\Sigma}(B, \Gamma) = (\neg\top \vee \neg b) \vee (\neg\bot \vee \neg b) \equiv_{\Gamma} \top \\
&\mathrm{SynMg}_{\Sigma}(C, \Gamma) = \bot \\
&\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Gamma} b \not\equiv_{\Gamma} \mathrm{SynMg}_{\Sigma}(B, \Gamma) \to \mathrm{SynMg}_{\Sigma}(C, \Gamma) \equiv_{\Gamma} \bot
\end{aligned}$$

Thus, we obtain $\mathit{Mod}_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) \not\subseteq \mathit{Mod}_{\Gamma}(\neg\, \mathrm{SynMg}_{\Sigma}(B, \Gamma))$.

Proposition 5.11. *Let Σ be a signature with two or more elements and let $\Gamma \subsetneq \Sigma$ be a strict subsignature. There are formulas $A, B, C \in \mathcal{L}_{\Sigma}$ with $A \equiv B \to C$ such that:*

$$\mathrm{SynMg}_{\Sigma}(A, \Gamma) \not\equiv \mathrm{SynMg}_{\Sigma}(B, \Gamma) \to \mathrm{SynMg}_{\Sigma}(C, \Gamma)$$

In summary, we showed that marginalization is not fully compatible with standard connectives of propositional logic both for finite and infinite signatures, whereby disjunction can be listed as the only mentionable exception (see Table 1).

6 Marginalization and Variable Forgetting for Sets of Formulas

In this section, we investigate and discuss marginalization and variable forgetting for finite and infinite sets of formulas and for finite and infinite signatures. For

Connective	Compatibility	Counterexample
$\vee$	$\equiv$ (Proposition 5.1)	fully compatible
$\wedge$	$\models$ (Proposition 5.2)	Proposition 5.3 / Example 5.4
$\neg$	$=\!\mid$ (Proposition 5.5)	Proposition 5.6 / Example 4
$\rightarrow$	$=\!\mid$ (Proposition 5.9)	Proposition 5.11 / Example 5.10

Table 1: Overview of the compatibility of syntactic marginalization, respectively variable forgetting, with connectives of propositional logic. Here, $\equiv$ stands for full compatibility; $\models$ expresses that marginalization of the full formula implies the formula obtained by marginalizing the components of the connective; analogously, $=\!\mid$ expresses that marginalization of the components of the connectives implies the marginalized formula.

that, we will use results obtained in Section 5. While some results seem rather straightforward to obtain technically, these results are not trivial because one might fall quickly into the trap of thinking that marginalization behaves very intuitively for sets of propositional formulas. In particular, we will see that one has to be careful about representation, e.g., syntactic structure, when performing marginalization or variable forgetting, respectively.

Element-Wise Marginalization for Sets. In order to lift syntactic marginalization of a formula to sets of formulas a natural choice is element-wise marginalization of each single formula.

Definition 6.1 (element-wise marginalization). *Let* $X \subseteq \mathcal{L}_\Sigma$ *be a set of formula and* $\Gamma \subseteq \Sigma$*. The* element-wise marginalization of X (from Σ) to Γ*, written* $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$*, is given by* $\mathrm{EWSynMg}_\Sigma(X, \Gamma) = \{\mathrm{SynMg}_\Sigma(B, \Gamma) \mid B \in X\}$.

One can see that $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$ is always well-defined, even for those cases where X, Σ or Γ are infinite. This is mainly due to the fact that $\mathrm{SynMg}_\Sigma(B, \Gamma)$ is a well-defined propositional formula for any formula B (cf. Section 3).

Similarly to Definition 6.1, we define a notion of element-wise variable forgetting for a set of formulas as $\mathrm{EWVarForget}(X, \Gamma) = \{\mathrm{VarForget}(B, \Gamma) \mid B \in X\}$. In Section 3, we have seen that variable forgetting and marginalization are corresponding operations on formulas, and from this correspondence we easily obtain that $\mathrm{EWSynMg}_\Sigma(X, \Gamma) = \mathrm{EWVarForget}(X, \Sigma \setminus \Gamma)$ holds. Because of this, in this section and in the following sections, we will take the viewpoint of (syntactic) marginalization.

Recall that the semantics for a set of formulas is given by intersection, which corresponds to conjunction in the case of finite sets of formulas. We will see in the following that due to Proposition 5.3 the notion of element-wise marginalization behaves already very unexpectedly on finite sets and does not seem to be an adequate way to define syntactic marginalization of sets of formulas.

Proposition 6.2. *Let Σ be a signature with three or more elements. There is a finite set of formulas $X \subseteq \mathcal{L}_\Sigma$ and a signature $\Gamma \subseteq \Sigma$ such that for every formula $A \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma X$ we obtain:*

$$\mathrm{EWSynMg}_\Sigma(X, \Gamma) \not\models_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)$$

Proposition 6.2 shows that element-wise marginalization of sets of formulas is not reducible to syntactic marginalization of an equivalent formula. However, the syntactic marginalization of a formula equivalent to a set of formulas is complete, in the sense that every logical consequence of the element-wise marginalization is also a logical consequence of the syntactic marginalization of a corresponding formula. The following proposition attests this observation.

Proposition 6.3. *Let $X \subseteq \mathcal{L}_\Sigma$ be an arbitrary set of formulas, $A \in \mathcal{L}_\Sigma$ be a formula and $\Gamma \subseteq \Sigma$. If $A \equiv_\Sigma X$, then* $\mathrm{SynMg}_\Sigma(A, \Gamma) \models_\Gamma \mathrm{EWSynMg}_\Sigma(X, \Gamma)$.

We continue by demonstrating how the incompatibility of conjunction with syntactic marginalization carries over to sets of formulas.

Example 6.4. Let $\Sigma = \{a, s, f\}$ be the signature from Example 5.4 and let $A = a \wedge s \wedge f$ and $A_1 = a \wedge (s \leftrightarrow f)$ and $A_2 = a \wedge f$ be the formulas from the same example. As shown before, for $\Gamma = \{a, s\}$ we have $\mathrm{SynMg}_\Sigma(A, \Gamma) \equiv_\Gamma a \wedge s$ and $\mathrm{SynMg}_\Sigma(A_1, \Gamma) \equiv_\Gamma \mathrm{SynMg}_\Sigma(A_2, \Gamma) \equiv_\Gamma a$. This renders $\mathrm{SynMg}_\Sigma(A, \Gamma)$ to be semantically different to $\mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \mathrm{SynMg}_\Sigma(A_2, \Gamma)$. We reproduce this result by using sets of formulas. Let $X = \{A_1, A_2\}$ be the set containing A_1 and A_2, which is equivalent to A, i.e., $X \equiv_\Sigma A$. Applying element-wise marginalization to X yields $\mathrm{EWSynMg}_\Sigma(X, \Gamma) = \{\mathrm{SynMg}_\Sigma(A_1, \Gamma), \mathrm{SynMg}_\Sigma(A_2, \Gamma)\} \equiv_\Gamma \{a\}$, which is semantically different from $\mathrm{SynMg}_\Sigma(A, \Gamma)$.

Infinite Signatures. For a set of formulas $X \subseteq \mathcal{L}_\Sigma$, we say that X *is finitely representable over* Σ if there is a formulas $A \in \mathcal{L}_\Sigma$ such that $X \equiv_\Sigma A$. Clearly, if Σ is finite, we have that X is finitely representable. But in general, not every set of formulas is finitely representable when the signature is infinite. This give rise to representational problems that carry over to syntactic marginalization as well.

Example 6.5. Let $\Sigma = \{a, a_1, a_2, a_3, \ldots\}$ be an infinite signature. We consider the set of formulas $X = \{a, \overline{a}a_1, \overline{a}a_2, , \overline{a}a_3, , \overline{a}a_4 \ldots\}$. First, note that X is inconsistent, i.e., $Mod_\Sigma(X) = \varnothing$. This is because the formula a is inconsistent with all other formulas $\overline{a} \wedge a_i$ in X. Clearly, X is finitely representable, e.g., by employing the formula $\bot$. Now let $\Gamma = \{a_1, a_2, \ldots\}$ be the subsignature of Σ which contains every atom of Σ except a. The syntactic marginalization of a to Γ is $\mathrm{SynMg}_\Sigma(a, \Gamma) = \top \vee \bot$ and the syntactic marginalization of each $\overline{a} \wedge a_i \in X$ to Γ is $\mathrm{SynMg}_\Sigma(\overline{a} \wedge a_i, \Gamma) = (\top \wedge a_i) \vee (\bot \wedge a_i)$. Consequently, we have $\mathrm{SynMg}_\Sigma(a, \Gamma) \equiv_\Gamma \top$ and $\mathrm{SynMg}_\Sigma(\overline{a}a_i, \Gamma) \equiv_\Gamma a_i$. Hence, we have $\mathrm{SynMg}_\Sigma(X, \Gamma) \equiv_\Gamma \{a_1, a_2, \ldots\}$, implying that $\mathrm{SynMg}_\Sigma(X, \Gamma) \equiv_\Gamma \{a_1, a_2, \ldots\}$ is not finitely representable over Γ.

The following proposition is an implication of the observation made in Example 6.5.

Proposition 6.6. *If Σ is infinite, then there is a set of formulas $X \subseteq \mathcal{L}_\Sigma$ and a subsignature Γ such that* $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$ *is not not finitely representable, even when X is finitely representable.*

Nevertheless, note that even for sets of formulas that are not finitely representable, we can always obtain a finite representation for marginalizations to finite signatures.

Proposition 6.7. *Let $X \subseteq \mathcal{L}_\Sigma$ be a set of formulas and let $\Gamma \subseteq \Sigma$ be a subsignature. If Γ is finite, then* $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$ *is finitely representable over Γ.*

This section provides evidence that it is not obvious how marginalization and variable forgetting can be implemented on representations of sets of formulas. For infinite signatures, we showed that in the basic case, where the target subsignature is finite, the existence of representations of the marginalization of sets of formulas is guaranteed.

7 Marginalization for Deductively Closed Sets of Formulas

We will now review syntactic marginalization for deductively closed sets of formulas. In particular, Proposition 6.2 does not apply to deductively closed sets, and we can show that marginalization is indeed a useful and adequate notion for deductively closed sets.

Proposition 7.1. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductive closed set of formulas, let $A \in \mathcal{L}_\Sigma$ be a formula and $\Gamma \subseteq \Sigma$. If $A \equiv_\Sigma X$, then* $\mathrm{EWSynMg}_\Sigma(X, \Gamma) \models_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)$.

From Propositions 6.3 and 7.1 we obtain the following central observation.

Corollary 7.2. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set of formulas and let $A \in \mathcal{L}_\Sigma$ be a formula. If $A \equiv_\Sigma X$, then for every $\Gamma \subseteq \Sigma$, we have:*

$$\mathrm{EWSynMg}_\Sigma(X, \Gamma) \equiv_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)$$

Clearly, Corollary 7.2 implies that the deductive closures of $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$ and $\mathrm{SynMg}_\Sigma(A, \Gamma)$ are the same, i.e.

$$Cn_\Gamma(\mathrm{EWSynMg}_\Sigma(X, \Gamma)) = Cn_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma)).$$

However, in general, element-wise marginalization of a deductively closed set to Γ does not yield a deductively closed set. This is because $\mathrm{EWSynMg}_\Sigma(X, \Gamma)$ does not contain all syntactic equivalent formulas. Thus, after element-wise application of marginalization we have to apply deductive closure to obtain a deductively closed set. This gives the rationale for the following notion of syntactic marginalization of deductively closed sets.

Definition 7.3. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set and $\Gamma \subseteq \Sigma$. The* syntactic marginalization of X from Σ to Γ, *written* $\mathrm{SynMg}_\Sigma(X, \Gamma)$, *is*

$$\mathrm{SynMg}_\Sigma(X, \Gamma) = Cn_\Gamma(\mathrm{EWSynMg}_\Sigma(X, \Gamma)) \ ,$$

the deductive closure of the element-wise marginalization of X to Γ.

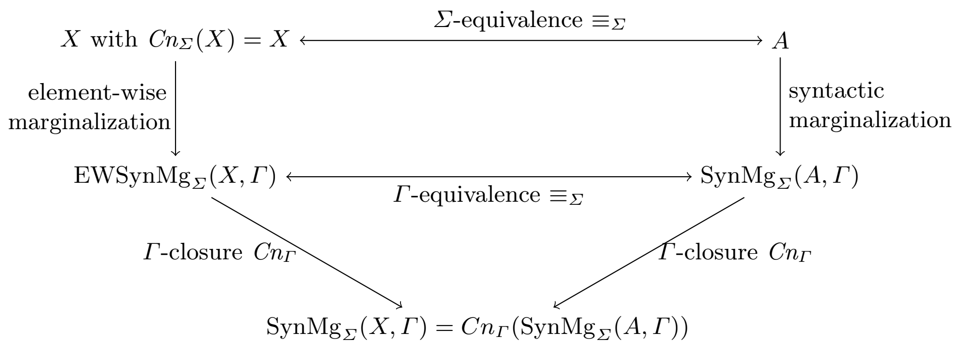

Fig. 5: Relations for a deductively closed set X and an equivalent formula A between element-wise marginalization, syntactic marginalization for a deductively closed set and syntactic marginalization for formula.

The following theorem describes that for a deductively closed set X, the syntactic marginalization of a representation of X and the syntactic marginalization of X comply with each other, semantically.

Theorem 7.4 (Representation Theorem for Marginalization). *For every deductively closed set $X \subseteq \mathcal{L}_\Sigma$ and every formula $A \in \mathcal{L}_\Sigma$ representing X, i.e., $X \equiv_\Sigma A$, and every $\Gamma \subseteq \Sigma$, we have:*

$$\mathrm{SynMg}_\Sigma(X, \Gamma) = Cn_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))$$

Figure 5 illustrates the compatibility between syntactic marginalization for formulas and syntactic marginalization for deductively closed sets. We continue with an example of syntactic marginalisation of deductively closed sets.

Example 7.5. Consider the signature $\Sigma = \{a, s, f\}$ from Example 5.4, and let A, A_1, A_2 as in Example 5.4. As explained in Example 6.4, we have that $X = \{A_1, A_2\}$ is semantically equivalent to A, i.e., we have $X \equiv_\Sigma A$. Furthermore, we already observed that of $\Gamma = \{a, s\}$ the element-wise marginalzation of X differs semantically from the syntactic marginalization of A, i.e., we have $\mathrm{EWSynMg}_\Sigma(X, \Gamma) \not\equiv_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)$. However, when considering $Cn(X)$, the situation is different. Theorem 7.4 guarantees that we have $\mathrm{SynMg}_\Sigma(Cn(X), \Gamma) \equiv_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)$. One reason for this last observation is that $Cn(X)$ does also contain A, consequently, $\mathrm{SynMg}_\Sigma(Cn(X), \Gamma)$ contains also $\mathrm{SynMg}_\Sigma(A, \Gamma)$.

Note that the concept introduced in Definition 7.3 and the result of Theorem 7.4 is of importance for potential application in many areas of knowledge representation. For instance, in syntax-splitting, belief revision and non-monotonic reasoning, deductively closed sets are often used to model agents' beliefs (also called belief sets). The incompatibility results from Section 5 and Section 6 indicate that marginalization and variable forgetting are not easily applicable techniques in the areas mentioned above, yet the results obtained in this section point out that this is not the case. Theorem 7.4 shows that formulas are an *adequate finite representation* for an agent's belief set when one wants to perform marginalization or variable forgetting on the agent's beliefs. Clearly, while this is always true in the case of finite signatures, for infinite signatures, this statement applies only to those belief sets that are finitely representable.

8 Applications of the Marginalization of Deductively Closed Sets

We consider some basic applications of the notion of syntactic marginalization.

Notions of forgetting. Another approach to forgetting is due to Delgrande [4] in which he proposes to understand forgetting of variables as a reduction to a sublanguage, i.e. $\mathit{forget}(X,\Gamma) = X \cap \mathcal{L}_{\Sigma\setminus\Gamma}$. We show that the approach by Delgrande complies with the notion of syntactic marginalization developed here.

Theorem 8.1 (Extended Representation Theorem for Marginalization). *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set and $A \in \mathcal{L}_\Sigma$ be a formula representing X, i.e. $X \equiv_\Sigma A$, then the following holds:*

$$\mathrm{SynMg}_\Sigma(X,\Gamma) = \mathit{Cn}_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) = X \cap \mathcal{L}_\Gamma$$

By Theorem 8.1, the extended representation theorem for deductively closed sets, we obtain different characterizations for syntactic marginalization for deductively closed sets and formulas equivalent to them.

Properties for Syntactic Marginalization. The following proposition summarizes useful properties of syntactic marginalization of deductively closed sets.

Proposition 8.2. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set and let $\Gamma \subseteq \Sigma$ and $\Gamma' \subseteq \Sigma$ be subsignatures of Σ. Syntactic marginalization satisfies the following properties:*

(Reduction) $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq \mathcal{L}_\Gamma$

(Inclusion) $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq X$

(Idempotency) $\mathrm{SynMg}_\Sigma(X,\Gamma) = \mathrm{SynMg}_\Sigma(\mathrm{SynMg}_\Sigma(X,\Gamma),\Gamma)$

(Monotonicity) If $\Gamma \subseteq \Gamma'$ holds, then $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq \mathrm{SynMg}_\Sigma(X,\Gamma')$

Observe that (Inclusion), (Idempotency) and (Monotonicity) from Proposition 8.2 are properties that are similar to the properties of an interior operator, which are dual to closure operators, i.e., Tarskian consequence relations[5].

Marginalization for Ordinal Conditional Functions. In knowledge representation and reasoning, the representation of an agent's epistemic state is often realized by ordinal conditional functions [23], also called ranking functions. A ranking function is a function $\kappa : \Omega_\Sigma \to \mathbb{N}_0$ such that there is at

[5] Closure operators satisfy (Monotonicity), (Idempotency) and (Extensitivity), i.e., $X \subseteq Cl(X)$.

least one interpretation ω with $\kappa(\omega) = 0$. The ranks assigned by such a κ are understood as degrees of implausibility, where the rank 0 stands for the most plausible rank. The belief set induced by a ranking function κ is the set of formulas $\mathrm{Bel}(\kappa) = \{\alpha \in \mathcal{L}_\Sigma \mid \{\omega \mid \kappa(\omega) = 0\} \subseteq \mathit{Mod}_\Sigma(\alpha)\}$, i.e., all Σ-formulas whose models are a superset of the most plausible models according to κ. The marginalization of κ to a subsignature Γ is defined as $\kappa|_\Gamma : \Omega_\Gamma \to \mathbb{N}_0$ with $\kappa|_\Gamma(\omega) = \min\{\kappa(\omega_\Sigma) \mid \omega_\Sigma \in \Omega_\Sigma, \omega_\Sigma^\Gamma = \omega\}$. Using Theorem 3.6 and Theorem 7.4, the following proposition relates $\mathrm{Bel}(\kappa)$ and $\mathrm{Bel}(\kappa|_\Gamma)$.

Proposition 8.3. *For every ranking function $\kappa : \Omega_\Sigma \to \mathbb{N}_0$ and every subsignature $\Gamma \subseteq \Omega$ we have:*

$$\mathrm{SynMg}_\Sigma(\mathrm{Bel}(\kappa), \Gamma) = \mathrm{Bel}(\kappa|_\Gamma)$$

Proposition 8.3 shows that the belief set of the marginalized ranking function coincides with the syntactically marginalized belief set of the original function. This nicely established relationship between the belief sets $\mathrm{Bel}(\kappa)$ and $\mathrm{Bel}(\kappa|_\Gamma)$ in Proposition 8.3 relies on the property that belief sets are deductively closed.

9 Conclusion

In this paper, we show that syntactic variable forgetting and semantic marginalization share the same basic technique, namely, aggregating the truth value of formulas over all possible interpretations of the atoms to be forgotten respectively to be suppressed. Semantically, marginalizing a possible world over an atom is the same as forgetting this variable from the complete conjunction that has this world as its only model. This can be successfully lifted to single formulas. Due to this close correspondence, we interpret variable forgetting as a syntactic marginalization operation.

However, we also point out clearly that one has to be careful when considering sets of formulas because variable forgetting applied to each of the formulas does not yield a result which is semantically equivalent to what one obtains after applying variable forgetting to the conjunction of the formulas. This is due to the fact that variable forgetting is not fully compatible with conjunction (and negation). Luckily for many scenarios considered in knowledge representation, semantic equivalence can be guaranteed here for deductively closed sets of formulas. In particular, this provides a justification for using formulas as a marginalization-compatible representation for deductively closed sets of formulas, as it is common, e.g., in belief revision theory [9]. Furthermore, we show that syntactic marginalization also complies with Delgrands' forgetting approach [4], and we provide some basic properties for syntactic marginalization.

We like to remark again that most properties and theorems for syntactic marginalization in this paper carry over to variable forgetting and model marginalization, as we showed that all these three operators yield semantically the same results (in a dual way). In future work, we will consider the compatibility of syntactic marginalization to different formalisms, like conditional logics or predicate logics, other kinds of forgetting, see, e.g., [2], and their axiomatics [21,11,12].

Acknowledgements. We thank the reviewers for their constructive and helpful comments. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) within the Priority Research Program "Intentional Forgetting in Organizations" (SPP 1921; grant BE 1700/10-1 awarded to Christoph Beierle and grant KE 1413/10-1 awarded to Gabriele Kern-Isberner).

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Supplementary Material

In the following, we give the proofs for the propositions and theorems given in the paper. First, we state a helpful observation regarding the expansion of signatures.

Observation 1 *Let Θ, Δ, and Σ be signatures with $\Theta \subseteq \Delta \subseteq \Sigma$ and let $A \in \mathcal{L}_\Sigma$ be a formula with $\mathrm{Sig}(A) \subseteq \Theta$. Furthermore, let ω be a Θ-model of A and let $\omega_\Delta \in \Omega_\Delta$ be an Δ-interpretation. If the restriction of ω_Δ to Θ coincide with ω, i.e., $\omega = \omega_\Delta^\Theta$, then ω_Δ is also a model of A, i.e., $\omega_\Delta \in Mod_\Delta(A)$.*

Theorem 3.6. *For every $A \in \mathcal{L}_\Sigma$ and $\Gamma \subseteq \Sigma$ the following holds:*

$$\begin{aligned}\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \Gamma) &= Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma)) \\ &= Mod_\Gamma(\mathrm{VarForget}(A, \Sigma \setminus \Gamma))\end{aligned}$$

Proof. By considering the definition of VarForget and SynMg we immediately obtain $\mathrm{VarForget}(A, \Sigma \setminus \Gamma) = \mathrm{SynMg}_\Sigma(A, \Gamma)$. We show that $\mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \Gamma) = Mod_\Gamma(\mathrm{VarForget}(A, \Sigma \setminus \Gamma))$ holds.

Considering the definition of VarForget yields that we have $\mathrm{VarForget}(A, \Sigma \setminus \Gamma) = \mathrm{VarForget}(A, \mathrm{Sig}(A) \setminus \Gamma)$. Consequently, we obtain that $\mathrm{Sig}(\mathrm{VarForget}(A, \Sigma \setminus \Gamma)) \subseteq \mathrm{Sig}(A) \cap \Gamma$ holds. However, because including additional atoms does not change modelhood (cf. Observation 1), we can safely focus on Σ-interpretations in the following to show the statement. From Lang et al. [15, Col. 5] we obtain the following observation:

$$Mod_\Sigma(\mathrm{VarForget}(A, \Sigma \setminus \Gamma)) = Mod_\Sigma(A) \cup \left\{ \omega_2 \in \Omega_\Sigma \;\middle|\; \begin{array}{c} \exists \omega_1 \in Mod_\Sigma(A) \\ \text{with } \omega_1^\Gamma = \omega_2^\Gamma \end{array} \right\} \quad (2)$$

Equation (2) implies that A is consistent (with respect to Σ-models) if and only if $\mathrm{VarForget}(A, \Sigma \setminus \Gamma)$ is consistent. Moreover, we conclude from Equation (2) that we have $\omega' \in Mod_\Gamma(\mathrm{VarForget}(A, \Sigma \setminus \Gamma))$ if and only if there exist an interpretation $\omega \in Mod_\Sigma(A)$ with $\omega' = \omega^\Gamma$. By considering the definition of ModMg, we obtain the desired result $Mod_\Gamma(\mathrm{VarForget}(A, \Sigma \setminus \Gamma)) = \mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \Gamma)$ from the last observation. □

Proposition 4.1. *For each propositional formula $A \in \mathcal{L}_\Sigma$ we have:*

$$\mathrm{Sig}_{\min}(A) = \{a \in \Sigma \mid \exists \omega_1, \omega_2 \in \Omega.\ \omega_1 \models_\Sigma A \textit{ and } \omega_2 \not\models_\Sigma A \textit{ and } \omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}\}$$

Proof. Let A be a propositional formula and let Y be the set

$$Y = \{a \in \Sigma \mid \exists \omega_1, \omega_2 \in \Omega_\Sigma.\ \omega_1 \models_\Sigma A \text{ and } \omega_2 \not\models_\Sigma A \text{ and } \omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}\}.$$

Note that $\mathrm{Sig}(A)$ is always finite, and consequently we have that $\mathrm{Sig}_{\min}(A)$ is also finite. We show that $\mathrm{Sig}_{\min}(A) = Y$ holds. At first, we consider the case of $A \equiv \top$ or $A \equiv \bot$. In this case, we directly obtain $Y = \varnothing = \mathrm{Sig}_{\min}(A)$. For the case of $A \not\equiv \bot$ and $A \not\equiv \top$, i.e. A is contingent, we show the equivalence $\mathrm{Sig}_{\min}(A) = Y$ by showing for each set the inclusion of one into another:

"$\subseteq$" To show that $Y \subseteq \mathrm{Sig}_{\min}(A)$ holds, we make a case distinction with respect to A. Towards a contradiction, assume $a \in Y$ and $a \notin \mathrm{Sig}_{\min}(A)$. Let B be a formula with $B \equiv_\Sigma A$ and such that the atom a does not appear in B. Because a does not appear in B, we have for each two interpretations $\omega_1, \omega_2 \in \Omega_\Sigma$ that $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$ implies $\omega_1 \in \mathit{Mod}_\Sigma(B)$ if and only if $\omega_2 \in \mathit{Mod}_\Sigma(B)$. Because we have $\mathit{Mod}_\Sigma(A) = \mathit{Mod}_\Sigma(B)$, this contradicts our assumption that $a \in Y$ holds. We obtain that every formula B which is equivalent to A makes use of each atom in Y and thus, $Y \subseteq \mathrm{Sig}_{\min}(A)$. A direct consequence is that Y is also a finite set.

"$\supseteq$" We show that $\mathrm{Sig}_{\min}(A) \subseteq Y$ holds. We define a formula $B \in \mathcal{L}_Y$ in disjunctive normal form as follows:

$$B = \bigvee_{B' \in \mathbb{B}} B' \qquad \text{where } \mathbb{B} = \left\{ \left(\bigwedge_{\substack{a \in Y \\ \omega \models a}} a \right) \wedge \left(\bigwedge_{\substack{a \in Y \\ \omega \not\models a}} \neg a \right) \,\middle|\, \omega \in \mathit{Mod}_\Sigma(A) \right\}$$

Because B uses only atoms from Y, we have $\mathrm{Sig}_{\min}(B) \subseteq \mathrm{Sig}(B) \subseteq Y$. Moreover, finiteness of Y implies finiteness of $\mathbb{B}$, hence B is a propositional formula. We show that B is equivalent to A by showing $\mathit{Mod}_\Sigma(B) = \mathit{Mod}_\Sigma(A)$. Clearly, by the construction of B, we obtain $\mathit{Mod}_\Sigma(A) \subseteq \mathit{Mod}_\Sigma(B)$. We show that the negation of $\mathit{Mod}_\Sigma(B) \subseteq \mathit{Mod}_\Sigma(A)$ leads to a contradiction. For this, we assume that there exists some $\omega_B \in \mathit{Mod}(B)$ with $\omega_B \notin \mathit{Mod}(A)$. By construction of B, there exists some $\omega_A \in \mathit{Mod}_\Sigma(A)$ such that $\omega_B^Y = \omega_A^Y$ holds. We make a case-distinction:

The case of $Y = \Sigma$. Clearly, $Y = \Sigma$ implies $\omega_A = \omega_B$ which is a contradiction to $\omega_B \notin \mathit{Mod}_\Sigma(A)$. Consequently, we have that $Y \neq \Sigma$ and $|\Sigma \setminus Y| \neq 0$ holds and that $\omega_B \neq \omega_A$ holds.

The case of $|\Sigma \setminus Y| = 1$. This case is impossible, because the existence of ω_B and ω_A would imply the contradiction stating that $a \in \Sigma \setminus Y$ and $a \in Y$ holds at the same time.

The case of $|\Sigma \setminus Y| > 1$. Let $X = \Sigma \setminus Y = \{a \in \Sigma \mid \omega_A^{\{a\}} \neq \omega_B^{\{a\}}\}$ be the set of all $a \in \Sigma$ such that $\omega_A^{\{a\}} \neq \omega_B^{\{a\}}$ holds. Consequently, X is non-empty. Now let $\Omega[Y] = \{\omega \in \Omega_\Sigma \mid \omega^Y = \omega_B^Y\}$ be the set of all interpretations that agree with ω_A regarding the valuation of the elements in Y. Because $\omega_A^Y = \omega_B^Y$ holds, we have $\omega_A, \omega_B \in \Omega[Y]$ and every interpretation in $\Omega[Y]$ agrees with ω_B regarding the valuation of the elements in Y. Furthermore, by construction of B, we obtain that $\Omega[Y] \subseteq \mathit{Mod}_\Sigma(B)$ holds. Now let $G = (\Omega[Y], E)$ be the undirected graph with $\Omega[Y]$ as vertices where two vertices $\omega_1, \omega_2 \in \Omega[Y]$ are neighbours in G if ω_1 and ω_2 disagree exactly in the valuation of one element from X, i.e., $\{\omega_1, \omega_2\} \in E$ if there exists exactly one $a \in X$ such that $\omega_1^{\{a\}} \neq \omega_2^{\{a\}}$ and $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$ holds. It is easy to see, that G is connected, i.e., for every two distinct vertices in G there is a path between these vertices in G. As ω_A, ω_B are vertices of G, there are models of A in G and non-models

of A in G. By considering G, one can observe that there exists $a \in X$ and there exist $\omega_1, \omega_2 \in \Omega[Y]$ such that
(a) $\omega_1 \in Mod_\Sigma(A)$ and $\omega_2 \notin Mod_\Sigma(A)$,
(b) $\omega_A^{\{a\}} = \omega_1^{\{a\}} \neq \omega_2^{\{a\}} = \omega_B^{\{a\}}$, and
(c) $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$
are satisfied. The latter holds because G is constructed in a way where neighbours disagree on the valuation of exactly one $a \in X$, and on the path from ω_A to ω_B in G, there must exist $\omega_1, \omega_2 \in \Omega[Y]$ with $\omega_1 \in Mod_\Sigma(A)$ and $\omega_2 \notin Mod_\Sigma(A)$. Existence of ω_1, ω_2 that satisfy (a)–(c) implies that $a \in Y$ holds. This is a contradiction to $a \in X = \Sigma \setminus Y$.

This shows B is equivalent to A. From $A \equiv B$ and $\mathrm{Sig}_{\min}(B) \subseteq Y$, we obtain $\mathrm{Sig}_{\min}(A) \subseteq Y$.

In summary, this shows that $\mathrm{Sig}_{\min}(A) = Y$ holds. □

Proposition 4.2. *Let Σ and Γ be signatures with $\Gamma \subseteq \Sigma$ and let $A \in \mathcal{L}_\Sigma$. The following statements hold:*

(a) We have that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A, \Gamma)) \subseteq \mathrm{Sig}_{\min}(A) \cap \Gamma$ holds.
(b) We have that $a \in \mathrm{Sig}_{\min}(A)$ if and only if $A \not\equiv_\Sigma \mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\})$ holds.
(c) If A is consistent, then $\Gamma \cap \mathrm{Sig}_{\min}(A) = \varnothing$ if and only if $\mathrm{SynMg}_\Sigma(A, \Gamma) \equiv_\Sigma \top$.

Proof. We consider the statements (a)–(c) step by step.

We start by showing that statement (a) holds.

Clearly, by definition of SynMg, we obtain $\mathrm{Sig}(\mathrm{SynMg}_\Sigma(A, \Gamma)) \subseteq \Gamma$. Because Sig is always a upper bound for the minimal set of atoms $\mathsf{Sig}_{\min}$, we directly obtain $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A, \Gamma)) \subseteq \mathrm{Sig}(\mathrm{SynMg}_\Sigma(A, \Gamma))$. It remains to show that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A, \Gamma)) \subseteq \mathrm{Sig}_{\min}(A)$ holds. There is a formula A' with $A' \equiv A$ and $\mathrm{Sig}(A') = \mathrm{Sig}_{\min}(A') = \mathrm{Sig}_{\min}(A)$. By considering the definition of SynMg we easily conclude that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A', \Gamma)) \subseteq \mathrm{Sig}(\mathrm{SynMg}_\Sigma(A', \Gamma))$ and $\mathrm{Sig}(\mathrm{SynMg}_\Sigma(A', \Gamma)) \subseteq \mathrm{Sig}_{\min}(A')$ holds. From Theorem 3.6 we obtain that $\mathrm{SynMg}_\Sigma(A, \Gamma) \equiv \mathrm{SynMg}_\Sigma(A', \Gamma)$ holds. By the observations shown above, we have $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A, \Gamma)) = \mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A', \Gamma))$ and $\mathrm{Sig}_{\min}(A) = \mathrm{Sig}_{\min}(A')$ and $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A', \Gamma)) \subseteq \mathrm{Sig}_{\min}(A')$. These observations together show that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_\Sigma(A, \Gamma)) \subseteq \mathrm{Sig}_{\min}(A)$ holds.

We show that statement (b) holds. We consider each direction of the statement independently.

"⇒"-direction. If $a \in \mathrm{Sig}_{\min}(A)$ holds, then according to Proposition 4.1, there are interpretations $\omega_1, \omega_2 \in \Omega_\Sigma$ such that $\omega_1 \models A$ and $\omega_2 \not\models A$ and $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$. Clear, we have that $\mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\}) \subseteq \mathrm{Sig}(A) \setminus \{a\}$ holds. Because we have $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$, we have $\omega_1^{\Sigma\setminus\{a\}} \models \mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\})$ if and only if we have $\omega_2^{\Sigma\setminus\{a\}} \models \mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\})$. Consequently, we either have $\omega_1, \omega_2 \in Mod_\Sigma(\mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\}))$ or we have $\omega_1, \omega_2 \notin Mod_\Sigma(\mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\}))$. Because $\omega_1 \in Mod_\Sigma(A)$ and $\omega_2 \notin Mod_\Sigma(A)$ holds, the last observation implies $A \not\equiv_\Sigma \mathrm{SynMg}_\Sigma(A, \Sigma \setminus \{a\})$.

"⇐"-direction. Assume that $A \not\equiv_{\Sigma} \mathrm{SynMg}_{\Sigma}(A, \Sigma \setminus \{a\})$ holds. Clearly, for each $\omega \in Mod_{\Sigma\setminus\{a\}}(\mathrm{SynMg}_{\Sigma}(A, \Sigma \setminus \{a\}))$ there are exactly two $\omega_1, \omega_2 \in \Omega_{\Sigma}$ with $\omega = \omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$ and $\omega_1(a) \neq \omega_2(a)$. By considering the definition of SynMg and by using Theorem 3.6, we obtain $Mod_{\Sigma}(A) \subseteq Mod_{\Sigma}(\mathrm{SynMg}_{\Sigma}(A, \Sigma\setminus\{a\}))$. From these observations and $A \not\equiv_{\Sigma} \mathrm{SynMg}_{\Sigma}(A, \Sigma\setminus\{a\})$, we conclude that $\omega_1, \omega_2 \in Mod_{\Sigma}(\mathrm{SynMg}_{\Sigma}(A, \Sigma \setminus \{a\}))$ and $\omega_1 \in Mod_{\Sigma}(A)$ and $\omega_2 \notin Mod_{\Sigma}(A)$ holds. Hence, from Proposition 4.1 we obtain that $a \in \mathrm{Sig}_{\min}(A)$ holds.

Next, we show that statement (c) holds. Let $A \in \mathcal{L}_{\Sigma}$ be a consistent formula. We consider each direction of the statement independently.

"⇒"-direction. Assume that $\Gamma \cap \mathrm{Sig}_{\min}(A) = \varnothing$ holds. From (a) we obtain that $\mathrm{Sig}_{\min}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) = \varnothing$ holds. Clearly, $\mathrm{Sig}_{\min}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) = \varnothing$ implies that $\mathrm{SynMg}_{\Sigma}(A, \Gamma)$ is either tautological or inconsistent. Theorem 3.6 yields that $\mathrm{SynMg}_{\Sigma}(A, \Gamma)$ is consistent exactly if A is consistent. Consequently, because A is consistent, we obtain the desired result that $\mathrm{SynMg}_{\Sigma}(A, \Gamma)$ is tautological.

"⇐"-direction. Assume that $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Sigma} \top$ holds. If A is tautological, the statement is trivial. We consider the case where A is not tautological yet consistent. For this case we have that $\mathrm{Sig}_{\min}(A) \neq \varnothing$. Towards a contradiction assume that $\Gamma \cap \mathrm{Sig}_{\min}(A) \neq \varnothing$ holds. Let a be an atom with $a \in \Gamma \cap \mathrm{Sig}_{\min}(A)$. From $a \in \mathrm{Sig}_{\min}(A)$ we obtain that there are $\omega_1, \omega_2 \in \Omega_{\Sigma}$ with $\omega_1^{\Sigma\setminus\{a\}} = \omega_2^{\Sigma\setminus\{a\}}$ and $\omega_1(a) \neq \omega_2(a)$ and $\omega_1 \models A$ and $\omega_2 \not\models A$. From $a \in \Gamma$, we obtain that $\omega_1^{\Gamma} \neq \omega_2^{\Gamma}$ holds. By invoking Theorem 3.6, we obtain $\omega_2^{\Gamma} \notin Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma))$ from $\omega_2 \not\models A$. Consequently, we also have $\omega_2 \notin Mod_{\Sigma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma))$. This last observation is a contradiction to $\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Sigma} \top$. □

Proposition 5.1. *For each $A \equiv_{\Sigma} A_1 \vee \ldots \vee A_n$ with $A, A_1, \ldots A_n \in \mathcal{L}_{\Sigma}$ and each $\Gamma \subseteq \Sigma$ the following holds:*

$$\mathrm{SynMg}_{\Sigma}(A, \Gamma) \equiv_{\Gamma} \mathrm{SynMg}_{\Sigma}(A_1, \Gamma) \vee \ldots \vee \mathrm{SynMg}_{\Sigma}(A_n, \Gamma)$$

Proof. The equivalence follows from the definition of variable forgetting, commutativity of $\vee$ and associativity of $\vee$. □

Proposition 5.2. *For each $A, A_1, \ldots A_n \in \mathcal{L}_{\Sigma}$ with $A \equiv_{\Sigma} A_1 \wedge \ldots \wedge A_n$ and for each $\Gamma \subseteq \Sigma$ the following holds:*

$$Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma)) \subseteq Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A_1, \Gamma) \wedge \ldots \wedge \mathrm{SynMg}_{\Sigma}(A_n, \Gamma)) \quad (3)$$

Proof. Observe that $A \equiv A_1 \wedge \ldots \wedge A_n$ if and only if $Mod_{\Sigma}(A) = Mod_{\Sigma}(A_1) \cap \ldots \cap Mod_{\Sigma}(A_n)$. Let ω' be an interpretation from $\mathrm{ModMg}_{\Sigma}(Mod_{\Sigma}(A), \Gamma)$. By Theorem 3.6 we obtain $\omega' \in Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A, \Gamma))$. By definition there exists $\omega \in Mod(A)$ such that $\omega' = \omega^{\Gamma}$. By our assumptions, $\omega \in Mod(A_i)$ holds for each $i \in \{1, \ldots, n\}$. We obtain $\omega' = \omega^{\Gamma} \in \mathrm{ModMg}_{\Sigma}(Mod(A_i), \Gamma)$ for each $i \in \{1, \ldots, n\}$. Employing Theorem 3.6 again yields $\omega' \in Mod_{\Gamma}(\mathrm{SynMg}_{\Sigma}(A_i, \Gamma)$ for each $i \in \{1, \ldots, n\}$. □

Proposition 5.3. *Let Σ be a signature with three or more elements. There exist $A, A_1, \ldots A_n \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma A_1 \wedge \ldots \wedge A_n$ and $\Gamma \subseteq \Sigma$ such that:*

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \ldots \wedge \mathrm{SynMg}_\Sigma(A_n, \Gamma)) \not\subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))$$

Proof. Consider the signature $\Sigma = \{a, b, c\}$ and $\Gamma = \{a, b\}$. Let $A = abc$ and $A_1 = (abc) \vee (a\overline{b}\overline{c})$ and $A_2 = (abc) \vee (a\overline{b}c)$. Clearly, the formula A is equivalent to $A_1 \wedge A_2$ with respect to Σ, i.e. $A \equiv_\Sigma A_1 \wedge A_2$. Syntactic marginalization of A yields $\mathrm{SynMg}_\Sigma(A, \Gamma) \equiv_\Gamma ab$. Moreover, we obtain $\mathrm{SynMg}_\Sigma(A_1, \Gamma) \equiv_\Gamma a$ and $\mathrm{SynMg}_\Sigma(A_2, \Gamma) \equiv_\Gamma a$. Consequently, we have $\mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \mathrm{SynMg}_\Sigma(A_2, \Gamma) \equiv_\Gamma a$. In summary, we obtain

$$\mathrm{SynMg}_\Sigma(A, \Gamma) \not\equiv_\Gamma \mathrm{SynMg}_\Sigma(A_1, \Gamma) \wedge \mathrm{SynMg}_\Sigma(A_2, \Gamma).$$

Because expansion of Σ by additional atoms does not change modelhood of any interpretation considered here (cf. Observation 1), the considered example carries over to any signature with three or more elements. □

Proposition 5.5. *For all $A, B \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma \neg B$ and for all $\Gamma \subseteq \Sigma$ we have:*

$$Mod_\Gamma(\neg(\mathrm{SynMg}_\Sigma(B, \Gamma))) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))$$

Proof. Employing Thm. 3.6, the claim holds due to the following chain of observations:

$$\begin{aligned}
Mod_\Gamma(\neg(\mathrm{SynMg}_\Sigma(B, \Gamma))) &= \Omega_\Gamma \setminus Mod_\Gamma(\mathrm{SynMg}_\Sigma(B, \Gamma)) \\
&= \Omega_\Gamma \setminus \mathrm{ModMg}_\Sigma(Mod_\Sigma(B), \Gamma) \\
&= \Omega_\Gamma \setminus \{\omega' \in \Omega_\Gamma \mid \exists \omega \in Mod_\Sigma(B) \text{ with } \omega' = \omega^\Gamma\} \\
&= \{\omega' \in \Omega_\Gamma \mid \forall \omega \in Mod_\Sigma(B) \text{ with } \omega' \neq \omega^\Gamma\} \\
&\subseteq \{\omega' \in \Omega_\Gamma \mid \exists \omega \in \Omega_\Sigma \setminus Mod_\Sigma(B) \text{ with } \omega' = \omega^\Gamma\} \\
&= \mathrm{ModMg}_\Sigma(\Omega_\Sigma \setminus Mod_\Sigma(B), \Gamma) \\
&= \mathrm{ModMg}_\Sigma(Mod_\Sigma(A), \Gamma) \\
&= Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma))
\end{aligned}$$

□

Proposition 5.6. *Let Σ be a signature with two or more elements and let $\Gamma \subsetneq \Sigma$ be a strict subsignature. There are formulas $A, B \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma \neg B$ such that:*

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma)) \not\subseteq Mod_\Gamma(\neg \mathrm{SynMg}_\Sigma(B, \Gamma))$$

Proof. Let $\Sigma = \{a, b, \ldots\}$ and let $\Gamma \subseteq \Sigma$ such that $a \notin \Gamma$ and $b \in \Gamma$. We choose the formulas $A = ab$ and $B = \overline{a} \vee \overline{b}$. Clearly, $A \equiv_\Sigma \neg B$ holds. Consider the following chain of observations:

$$\begin{aligned}
\mathrm{SynMg}_\Sigma(A, \Gamma) &= (\top b) \vee (\bot b) \equiv_\Gamma b \\
\mathrm{SynMg}_\Sigma(B, \Gamma) &= (\neg\top \vee \overline{b}) \vee (\neg\bot \vee \overline{b}) \equiv_\Gamma \top \\
\neg \mathrm{SynMg}_\Sigma(B, \Gamma) &\equiv_\Gamma \bot \not\equiv_\Gamma b \equiv_\Gamma \mathrm{SynMg}_\Sigma(A, \Gamma)
\end{aligned}$$

Thus, we obtain $Mod_\Gamma(\mathrm{SynMg}_\Sigma(A, \Gamma)) \not\subseteq Mod_\Gamma(\neg \mathrm{SynMg}_\Sigma(B, \Gamma))$. □

Proposition 5.9. *For each $A, B, C \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma B \to C$ and for each $\Gamma \subseteq \Sigma$ the following holds:*

$$\mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma) \to \mathrm{SynMg}_\Sigma(C,\Gamma)) \subseteq \mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$$

Proof. Let ω be a model of $\mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma) \to \mathrm{SynMg}_\Sigma(C,\Gamma))$, i.e., we have either $\omega \in \mathit{Mod}_\Gamma(\neg\,\mathrm{SynMg}_\Sigma(B,\Gamma))$ or we have $\omega \in \mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(C,\Gamma))$. We make a case distinction:

The "$\omega \in \mathit{Mod}_\Gamma(\neg\,\mathrm{SynMg}_\Sigma(B,\Gamma))$" case. First, recall that due to Proposition 5.5 it holds that $\mathit{Mod}_\Gamma(\neg\,\mathrm{SynMg}_\Sigma(B,\Gamma)) \subseteq \mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(\neg B,\Gamma))$. From Theorem 3.6 we obtain that there is some model $\omega_{\neg B} \in \mathit{Mod}_\Sigma(\neg B)$ such that $\omega = \omega_{\neg B}^\Gamma$. Because we have $A \equiv B \to C \equiv \neg B \vee C$, we conclude that $\omega_{\neg B}$ is also a model of $\mathit{Mod}_\Sigma(A)$. Consequently, Theorem 3.6 implies that $\omega_{\neg B}^\Gamma$ is also a model of $\mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$.

The "$\omega \in \mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(C,\Gamma))$" case. From Theorem 3.6 we obtain that there is some model $\omega_C \in \mathit{Mod}_\Sigma(\neg C)$ such that $\omega = \omega_C^\Gamma$. Because we have $A \equiv B \to C \equiv \neg B \vee C$, we conclude that ω_C is also a model of $\mathit{Mod}_\Sigma(A)$. Consequently, Theorem 3.6 implies that ω_C^Γ is also a model of $\mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$.

This completes the proof. □

Proposition 5.11. *Let Σ be a signature with two or more elements and let $\Gamma \subsetneq \Sigma$ be a strict subsignature. There are formulas $A, B, C \in \mathcal{L}_\Sigma$ with $A \equiv B \to C$ such that:*

$$\mathrm{SynMg}_\Sigma(A,\Gamma) \not\equiv \mathrm{SynMg}_\Sigma(B,\Gamma) \to \mathrm{SynMg}_\Sigma(C,\Gamma)$$

Proof. Let $\Sigma = \{a, b, \ldots\}$ and let be $\Gamma \subseteq \Sigma$ a subsignature such that $a \notin \Gamma$ and $b \in \Gamma$. We choose the formulas $A = a \wedge b$ and $B = \overline{a} \vee \overline{b}$ and $C = \bot$. One can observe easily that $A \equiv_\Sigma B \to C = \neg B \vee C$ holds.

$$\begin{aligned}
&\mathrm{SynMg}_\Sigma(A,\Gamma) = (\top \wedge b) \vee (\bot \wedge b) \equiv_\Gamma b \\
&\mathrm{SynMg}_\Sigma(B,\Gamma) = (\neg\top \vee \overline{b}) \vee (\neg\bot \vee \overline{b}) \equiv_\Gamma \top \\
&\mathrm{SynMg}_\Sigma(C,\Gamma) = \bot \\
&\mathrm{SynMg}_\Sigma(A,\Gamma) \equiv_\Gamma b \not\equiv_\Gamma \mathrm{SynMg}_\Sigma(B,\Gamma) \to \mathrm{SynMg}_\Sigma(C,\Gamma) \equiv_\Gamma \bot
\end{aligned}$$

Thus, we obtain $\mathit{Mod}_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) \not\subseteq \mathit{Mod}_\Gamma(\neg\,\mathrm{SynMg}_\Sigma(B,\Gamma))$. □

Proposition 6.2. *Let Σ be a signature with three or more elements. There is a finite set of formulas $X \subseteq \mathcal{L}_\Sigma$ and a signature $\Gamma \subseteq \Sigma$ such that for every formula $A \in \mathcal{L}_\Sigma$ with $A \equiv_\Sigma X$ we obtain:*

$$\mathrm{EWSynMg}_\Sigma(X,\Gamma) \not\models_\Gamma \mathrm{SynMg}_\Sigma(A,\Gamma)$$

Proof. Let $X = \{A_1, A_2\}$, where A_1 and A_2 are the formulas from the proof of Proposition 5.3. Note that by Theorem 3.6 the specific syntactic form of A does not matter for the semantic compatibility between marginalization and variable forgetting. Thus, we choose A as in the proof of Proposition 5.3. As a direct consequence of the proof of Proposition 5.3 we obtain the desired result. □

Proposition 6.3. *Let $X \subseteq \mathcal{L}_\Sigma$ be an arbitrary set of formulas, $A \in \mathcal{L}_\Sigma$ be a formula and $\Gamma \subseteq \Sigma$. If $A \equiv_\Sigma X$, then* $\mathrm{SynMg}_\Sigma(A,\Gamma) \models_\Gamma \mathrm{EWSynMg}_\Sigma(X,\Gamma)$.

Proof. By using Theorem 3.6, we obtain:

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) = \{\omega^\Gamma \mid \omega \in Mod_\Sigma(A)\} = \{\omega^\Gamma \mid \omega \in Mod_\Sigma(X)\}$$

For EWSynMg, again by using Theorem 3.6, we obtain the following equivalence:

$$\begin{aligned} Mod_\Gamma(\mathrm{EWSynMg}_\Sigma(X,\Gamma)) &= \bigcap_{B\in X} Mod_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma)) \\ &= \bigcap_{B\in X} \{\omega^\Gamma \mid \omega \in Mod_\Sigma(B)\} \end{aligned}$$

Because we have $Mod_\Sigma(X) \subseteq Mod_\Sigma(B)$ for each $B \in X$, we obtain the following inclusion:

$$\{\omega^\Gamma \mid \omega \in Mod_\Sigma(X)\} \subseteq \bigcap_{B\in X} \{\omega^\Gamma \mid \omega \in Mod_\Sigma(B)\}$$

The last inclusion implies the statement to be shown. □

Proposition 6.7. *Let $X \subseteq \mathcal{L}_\Sigma$ be a set of formulas and let $\Gamma \subseteq \Sigma$ be a subsignature. If Γ is finite, then* $\mathrm{EWSynMg}_\Sigma(X,\Gamma)$ *is finitely representable over Γ.*

Proof (sketch). Assume that Γ is finite. Because Γ is finite, there are only finitely many $\equiv_\Gamma$-equivalent formulas in $\mathcal{L}_\Gamma$. Clearly, $\mathrm{EWSynMg}_\Sigma(X,\Gamma)$ is a subset of $\mathcal{L}_\Gamma$, and thus, $\mathrm{EWSynMg}_\Sigma(X,\Gamma) \equiv_\Gamma A$ contains only finitely many $\equiv_\Gamma$-equivalent formulas. For each $\equiv_\Gamma$-equivalence class $[A_i]_{\equiv_\Gamma}$ which shares a formula with $\mathrm{EWSynMg}_\Sigma(X,\Gamma)$, i.e., $[A_i]_{\equiv_\Gamma} \cap \mathrm{EWSynMg}_\Sigma(X,\Gamma)$ is non-empty, select a unique representative formula A_i. The formula A is the conjunction of all these finitely many A_i. One can easily check that $\mathrm{EWSynMg}_\Sigma(X,\Gamma) \equiv_\Gamma A$ holds.

Proposition 7.1. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductive closed set of formulas, let $A \in \mathcal{L}_\Sigma$ be a formula and $\Gamma \subseteq \Sigma$. If $A \equiv_\Sigma X$, then* $\mathrm{EWSynMg}_\Sigma(X,\Gamma) \models_\Gamma \mathrm{SynMg}_\Sigma(A,\Gamma)$.

Proof. Because of $A \equiv_\Sigma X$ and X is deductively closed, we also have $A \in X$. Consequently, we have $\mathrm{SynMg}_\Sigma(A,\Gamma) \in \mathrm{EWSynMg}_\Sigma(X,\Gamma)$. It follows directly that $Mod_\Gamma(\mathrm{EWSynMg}_\Sigma(X,\Gamma)) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$ holds. □

Theorem 7.4. *For every deductively closed set $X \subseteq \mathcal{L}_\Sigma$ and every formula $A \in \mathcal{L}_\Sigma$ representing X, i.e., $X \equiv_\Sigma A$, and every $\Gamma \subseteq \Sigma$, we have:*

$$\mathrm{SynMg}_\Sigma(X,\Gamma) = Cn_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$$

Proof. Because X is deductively closed and we have $A \equiv_\Sigma X$, we also have that $A \in X$ holds. Consequently, we also have that $\mathrm{SynMg}_\Sigma(A,\Gamma) \in \mathrm{SynMg}_\Sigma(X,\Gamma)$ holds. Hence, we also have $Mod_\Gamma(\mathrm{SynMg}_\Sigma(X,\Gamma)) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$.

We will show by contradiction that $Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(X,\Gamma))$ holds. For that, assume that there is some $\omega \in \Omega_\Gamma$ with

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) \text{ and } \omega \notin Mod_\Gamma(\mathrm{SynMg}_\Sigma(X,\Gamma))\ .$$

Such an interpretation ω exists only if there is some formula $B \in X$ with

$$\mathrm{SynMg}_\Sigma(B,\Gamma) \in Mod_\Gamma(\mathrm{SynMg}_\Sigma(X,\Gamma))$$

such that $\omega \notin Mod_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma))$ holds. Using $B \in X$ and $X \equiv_\Sigma A$, we obtain that $Mod_\Sigma(A) \subseteq Mod_\Sigma(B)$ holds. By using Collorary 3.7 (b), we obtain

$$Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) \subseteq Mod_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma))$$

from $Mod_\Sigma(A) \subseteq Mod_\Sigma(B)$. Hence we have that $\omega \notin Mod_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma))$ and $\omega \in Mod_\Gamma(\mathrm{SynMg}_\Sigma(B,\Gamma))$ hold at the same time, which is a contradiction. We obtain that $Mod_\Gamma(\mathrm{SynMg}_\Sigma(X,\Gamma)) = Mod_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$ holds. The statment $\mathrm{SynMg}_\Sigma(X,\Gamma) = Cn_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma))$ is a consequence of the last observation. □

Theorem 8.1. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set and $A \in \mathcal{L}_\Sigma$ be a formula representing X, i.e. $X \equiv_\Sigma A$, then the following holds:*

$$\mathrm{SynMg}_\Sigma(X,\Gamma) = Cn_\Gamma(\mathrm{SynMg}_\Sigma(A,\Gamma)) = X \cap \mathcal{L}_\Gamma$$

Proof. Proposition 7.4 provides the first equality. We show the second equality by proving $\mathrm{SynMg}_\Sigma(X,\Gamma) = X \cap \mathcal{L}_\Gamma$.

Let A a formula such that $A \equiv_\Sigma X$. By Proposition 7.4 we obtain $\mathrm{SynMg}_\Sigma(A,\Gamma) \equiv_\Gamma \mathrm{SynMg}_\Sigma(X,\Gamma)$. Now observe that $X \cap \mathcal{L}_\Gamma = \{B \mid B \in \mathcal{L}_\Gamma, B \in X\}$. Clearly, we have for every $B \in \mathcal{L}_\Gamma$ that $B \in X$ if and only if $A \models_\Sigma B$. A direct consequence is $\mathrm{SynMg}_\Sigma(A,\Gamma) \models_\Gamma B$ and therefore we obtain $\mathrm{SynMg}_\Sigma(A,\Gamma) \models_\Gamma X \cap \mathcal{L}_\Gamma$. Consequently, we obtain $\mathrm{SynMg}_\Sigma(X,\Gamma) \models_\Gamma X \cap \mathcal{L}_\Gamma$.

Towards a contradiction suppose there is some $\omega' \in \Omega_\Gamma$ with $\omega' \models_\Gamma X \cap \mathcal{L}_\Gamma$ and $\omega' \not\models_\Gamma \mathrm{SynMg}_\Sigma(A,\Gamma)$. This implies that $\omega' \models_\Gamma B$ for every $B \in X \cap \mathcal{L}_\Gamma$. By definition, there exist some $\omega \in \Omega_\Sigma$ such that $\omega' = \omega^\Gamma$ and $\omega \models_\Sigma X \cap \mathcal{L}_\Gamma$. As consequence, we obtain the contradiction $\omega^\Gamma \models_\Gamma \mathrm{SynMg}_\Sigma(A,\Gamma)$. Thus, we have shown $\mathrm{SynMg}_\Sigma(X,\Gamma) \equiv_\Gamma X \cap \mathcal{L}_\Gamma$. Note that $\mathrm{SynMg}_\Sigma(X,\Gamma)$ is deductively closed. To see that $X \cap \mathcal{L}_\Gamma$ is likewise deductively closed, observe as first step that $X \cap \mathcal{L}_\Gamma$ is closed under logical equivalence. Moreover, observe that every logical consequence B of $X \cap \mathcal{L}_\Gamma$ is also an element of X. Both observations together complete the proof. □

Proposition 8.2. *Let $X \subseteq \mathcal{L}_\Sigma$ be a deductively closed set and let $\Gamma \subseteq \Sigma$ and $\Gamma' \subseteq \Sigma$ be subsignatures of Σ. Syntactic marginalization satisfies the following properties:*

(Reduction) $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq \mathcal{L}_\Gamma$

(Inclusion) $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq X$

(Idempotency) $\mathrm{SynMg}_\Sigma(X,\Gamma) = \mathrm{SynMg}_\Sigma(\mathrm{SynMg}_\Sigma(X,\Gamma),\Gamma)$

(Monotonicity) If $\Gamma \subseteq \Gamma'$ holds, then $\mathrm{SynMg}_\Sigma(X,\Gamma) \subseteq \mathrm{SynMg}_\Sigma(X,\Gamma')$

Proof. Satisfaction of (Success) and (Inclusion) is immediate due to Theorem 8.1. Moreover, by Theorem 8.1 we obtain (Idempotency) from the set-theoretic idempotency of intersection in $X \cap \mathcal{L}_\Gamma = (X \cap \mathcal{L}_\Gamma) \cap \mathcal{L}_\Gamma$. From $\Gamma \subseteq \Gamma'$ we obtain $\mathcal{L}_\Gamma \subseteq \mathcal{L}_{\Gamma'}$, which carries over to $X \cap \mathcal{L}_\Gamma \subseteq X \cap \mathcal{L}_{\Gamma'}$. By using Theorem 8.1, we obtain (Monotonicity) from $X \cap \mathcal{L}_\Gamma \subseteq X \cap \mathcal{L}_{\Gamma'}$. □

Proposition 8.3. *For every ranking function $\kappa : \Omega_\Sigma \to \mathbb{N}_0$ and every subsignature $\Gamma \subseteq \Omega$ we have:*

$$\mathrm{SynMg}_\Sigma(\mathrm{Bel}(\kappa), \Gamma) = \mathrm{Bel}(\kappa|_\Gamma)$$

Proof. Note that $\mathrm{Bel}(\kappa) \subseteq \mathcal{L}_\Sigma$ and $\mathrm{Bel}(\kappa|_\Gamma) \subseteq \mathcal{L}_\Gamma$ are deductively closed sets of formulas. Because of that and because of Theorem 3.6 and Theorem 7.4 it is sufficient to show that

$$\mathrm{ModMg}_\Sigma(\mathit{Mod}_\Sigma(\mathrm{Bel}(\kappa)), \Gamma) = \mathit{Mod}_\Gamma(\mathrm{Bel}(\kappa|_\Gamma)) \tag{4}$$

holds. Observe that for the marginalization of ranking functions, it holds that:

$$\begin{aligned} \mathit{Mod}_\Gamma(\mathrm{Bel}(\kappa|_\Gamma)) &= \{\omega' \in \Omega_\Gamma \mid \kappa|_\Gamma(\omega') = 0\} \\ &= \{\omega^\Gamma \mid \omega \in \Omega_\Sigma \text{ and } \kappa(\omega) = 0\} \end{aligned} \tag{5}$$

From considering the definition of ModMg and $\mathrm{Bel}(\kappa)$ we obtain the following equivalences:

$$\begin{aligned} \mathrm{ModMg}_\Sigma(\mathit{Mod}_\Sigma(\mathrm{Bel}(\kappa)), \Gamma) &= \{\omega^\Gamma \mid \omega \in \mathit{Mod}_\Sigma(\mathrm{Bel}(\kappa))\} \\ &= \{\omega^\Gamma \mid \omega \in \Omega_\Sigma \text{ and } \kappa(\omega) = 0\} \end{aligned} \tag{6}$$

From Equation (5) and Equation (6), we obtain Equation (4) which was to be shown. □